Lecture 36 Dec 2, 2005

Note Title 11/30/2005

Last time: 1) We defined k-manifolds

2) we defined exterior derivative d of a k-form.

3) we stated (generalized) Stokes theorem:

$$(*) \quad \int_{\partial M} \omega = \int_M d\omega$$

[I am changing the notation slightly:
I am writing $\int_M d\omega$ instead of $\int\!\!\cdots\!\!\int_M d\omega$]

4) We checked that when $M \subseteq \mathbb{R}^2$ is a domain, $(*)$ amounts to Green's theorem.

5) We checked that when $M \subseteq \mathbb{R}^3$ is a domain $(*)$ is Divergence (Gauss) theorem.

Example

Compute $\displaystyle\int_\Sigma x^2y\, dy\wedge dz + 3y^2\, dz\wedge dx - 2xz^2\, dx\wedge dy$

where Σ is the surface of the unit cube $Q = \left\{(x,y,z) \,\middle|\, \begin{matrix} 0\le x\le 1 \\ 0 \le y \le 1 \\ 0\le z\le 1\end{matrix}\right\}$

Solution By Stokes,

$$\int_\Sigma (\cdots) = \int_Q d(\quad) = \int_Q d(x^2y)\wedge dy\wedge dz + d(3y^2) dz\wedge dx$$

$$- d(2xz^2)\wedge dx\wedge dy =$$

$$= \int_Q \Big(2xy\, dx\wedge dy\wedge dz + \overset{=0}{\cancel{x^2 dy\wedge dy\wedge dz}} + 6y\, dy\wedge dz\wedge dx -$$

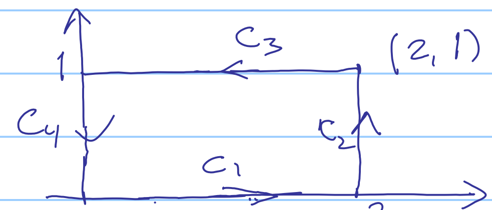
$$- \overset{=0}{\cancel{2z^2 dx\wedge dx\wedge dy}} - 4xz\, dz\wedge dx\wedge dy\Big) =$$

$$= \int_Q (2xy + 6y - 4xz)\, dx\wedge dy\wedge dz = \int_0^1\!\!\int_0^1\!\!\int_0^1 (6xy + 6y - 4xz)\, dx\,dy\,dz$$

$$= \int_0^1\!\!\int_0^1 (6xyz + 6y - 2xz^2)\Big|_0^1 dx\,dy = \int_0^1\!\!\int_0^1 (6xy + 6y - 2x)\, dx\,dy =$$

$$= \int_0^1 (3xy^2 + 3y^2 - 2xy)\Big|_0^1 dx = \int_0^1 (3x - 2x + 3)\, dx = \tfrac{1}{2} +$$

Example Let C be the boundary of the rectangle



Compute $\int_C (-x^2y\,dx + x^2y^2dy)$.

Solution 1 (direct computation)

$$\int_C (\ldots) = \int_{C_1}(\ldots) + \int_{C_2}(\ldots) + \int_{C_3}(\ldots) + \int_{C_4}(\ldots)$$

$C_1 = \{(t,0) \mid 0 \le t \le 2\}$

$C_2 = \{(2,t) \mid 0 \le t \le 1\}$

$C_3 = \{(2-t,1) \mid 0 \le t \le 2\}$

$C_4 = \{(1-t,0) \mid 0 \le t \le 1\}$.

$$\int_{C_1} -x^2y\,dx + x\,y^2dy = \int_{[0,2]} -t^2\cdot 0\,dt + t\cdot 0^2 d0 = 0$$

$$\int_{C_2} -x^2y\,dx + x\,y^2dy = \int_{[0,1]} -2^2t\,\overbrace{d(2)}^{=0} + 2t^2dt = \int_0^1 2t^2dt = \frac{2}{3}t^3\Big|_0^1 = 2/3.$$

$$\int_{C_3} -x^2y\,dx + x\,y^2dy = \int_{[0,2]} -(2-t)^2\,1\,d(2-t) + \underbrace{(2-t)\,1\,d(1)}_{=0} = -\frac{(2-t)^3}{3}\Big|_0^2 = -0 + \frac{8}{3}$$

$$\int_{C_4} -x^2y\,dx + x\,y^2dy = \int_{[0,1]} -(1-t)^2\cdot 0\,d(1-t) + (1-t)\cdot 0\,d(0) = 0$$

Solution 2 (Stokes)

$$\int_C (-x^2y\,dx + xy^2dy) = \iint_{\square} (d(-x^2y)\wedge dx + d(xy^2)\wedge dy =$$

$$= \iint_{\square} -x^2\,dy\wedge dx + y^2\,dx\wedge dy = \int_0^2\int_0^1 (x^2+y^2)\,dx\,dy = \int_0^2 \left(\frac{x^3}{3} + xy^2\right)\Big|_0^1 dy = \int_0^2 (y^2 + 1/3)\,dy = \frac{y^3}{3} + \frac{y}{3}\Big|_0^2 =$$

$$= \frac{2}{3} + \frac{8}{3} \quad \checkmark.$$

Recall the vector calculus $\leftrightarrow$ differential forms dictionary:

$$\int_C (F_1 \vec{e}_1 + \cdots + F_n \vec{e}_n) \cdot ds = \int_C F_1\,dx_1 + \cdots + F_n\,dx_n$$

(works in $\mathbb{R}^n$ for any n)

$$\iint_\Sigma (F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}) \cdot d\vec{S} = \iint_\Sigma F_1\,dy \wedge dz + F_2\,dz \wedge dx + F_3\,dx \wedge dy$$

[works only in $\mathbb{R}^3$].

$$\iiint_D f\,dV = \iiint_D f\,dx \wedge dy \wedge dz \quad \begin{bmatrix} D \text{ region} \\ \text{in } \mathbb{R}^3 \end{bmatrix}$$

The above is a piece of the dictionary:

functions f	$\longleftrightarrow$	0-forms f
$\nabla \downarrow$		$\downarrow d$
vector fields $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$	$\longleftrightarrow$	1-forms $\alpha_{\vec{F}} = F_1\,dx + F_2\,dy + F_3\,dz$
$\nabla\times \downarrow$		$\downarrow d$
vector fields $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$		2-forms $\omega_{\vec{F}} = F_1\,dy \wedge dz + F_2\,dz \wedge dx + F_3\,dx \wedge dy$
$\nabla\cdot \downarrow$		$\downarrow d$
functions f	$\longleftrightarrow$	3-forms $f\,dx \wedge dy \wedge dz$

$$\nabla \times (\nabla f) = 0 \quad \longleftrightarrow \quad d(df) = 0$$

$$\nabla \cdot (\nabla \times \vec{F}) = 0 \quad \Longleftrightarrow \quad d(d\alpha_{\vec{F}}) = 0$$

Theorem For any k-form ω

$$d(d\omega) = 0.$$

Reason: mixed partials commute.

Recall If D is a simply connected region in $\mathbb{R}^3$, $\vec{F}$ is a vector field and $\boxed{\nabla \times \vec{F} = 0, \text{ then } \vec{F} = \nabla f}$ for some function f.

Similarly, $\boxed{\text{if } \nabla \cdot \vec{F} = 0 \text{ then } \vec{F} = \nabla \times \vec{G}}$ for some vector field G.

These are two glimpses of one theorem (Poincaré):

> If $D \subseteq \mathbb{R}^n$ is a simply connected region and ω a k-form on D with $d\omega = 0$, then there is a $k-1$ form η so that
> $$d\eta = \omega.$$

Example Compute the integral of $x\,dx + y\,dy$ over the ellipse $C: \frac{x^2}{2^2} + \frac{y^2}{3^2} = 1.$

Solution $d(x\,dx + y\,dy) = dx \wedge dx + dy \wedge dy = 0.$

So there is a function f so that $df = x\,dx + y\,dy$. Now

$$\int_C x\,dx + y\,dy = \int_C df = \int_{\partial C} f = 0$$

since C is closed.