Lecture 37 December 5, 2005

Note Title 12/4/2005

Recall the dictionary:

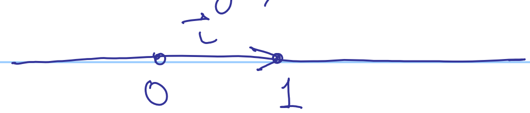
functions f	$\longleftrightarrow$	0-forms f
$\nabla \downarrow$		$\downarrow d$
vector fields $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$	$\longleftrightarrow$	1-forms $\alpha_{\vec{F}} = F_1\,dx + F_2\,dy + F_3\,dz$
$\nabla\times \downarrow$		$\downarrow d$
vector fields $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$		2-forms $\omega_{\vec{F}} = F_1\,dy\wedge dz + F_2\,dz\wedge dx + F_3\,dx\wedge dy$
$\nabla\cdot \downarrow$		$\downarrow d$
functions f	$\longleftrightarrow$	3-forms $f\,dx\wedge dy\wedge dz$

Today we straighten out something I swept under the rug: orientations.

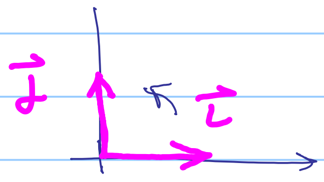
Let's start at the beginning: $\mathbb{R}^1$: ————

Its standard orientation is ⟶

Equivalently, it's determined by the vector $\vec{i} = (1)$

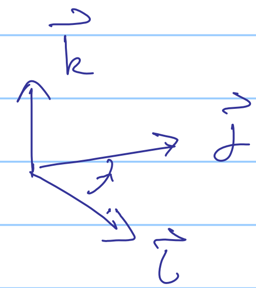


The standard orientation of $\mathbb{R}^2$ is given by an <u>ordered</u> pair of vectors $(\vec{i}, \vec{j})$:



If you like, $\mathbb{R}^2$ is oriented conterclockwise.

The standard orientation of $\mathbb{R}^3$ is given by the right hand rule:



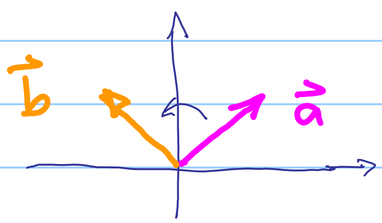
Or, by the triple

$(\vec{i}, \vec{j}, \vec{k})$

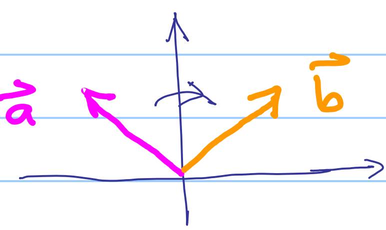
in this order.

Let's go back to $\mathbb{R}^2$:

$$(\underset{\vec{a}}{(1,1)}, \underset{\vec{b}}{(-1,1)})$$ give $\mathbb{R}^2$ positive orientation:



But $$(\underset{\vec{a}}{(-1,1)}, \underset{\vec{b}}{(1,1)})$$ give $\mathbb{R}^2$ the opposite orientation:



How can we tell?

$$\det\left(\vec{i} \,\middle|\, \vec{j}\right) = \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = +1 > 0$$

$$\det\left(\vec{a} \,\middle|\, \vec{b}\right) = \det\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = 2 > 0$$

$$\det\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = -2 < 0.$$

<u>Note</u> $$\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \det\begin{pmatrix} dx(\vec{i}) & dx(\vec{j}) \\ dy(\vec{i}) & dy(\vec{j}) \end{pmatrix} =$$

$$= dx \wedge dy\,(\vec{i}, \vec{j}).$$

Similarly $\det(\vec{a}\,|\,\vec{b}) = dx\wedge dy\,(\vec{a},\vec{b})$

„So we can translate:
$(\vec{a},\vec{b})$ gives $\mathbb{R}^2$ positive orientation"
into:
" $dx\wedge dy\,(\vec{a},\vec{b}) > 0$. "

In other words, it's $dx\wedge dy$ that orients $\mathbb{R}^2$.
$(-dx\wedge dy = dy\wedge dx$ gives $\mathbb{R}^2$ the opposite orientation$)$

Now let's examine $\mathbb{R}^3$:
three vectors $(\vec{a},\vec{b},\vec{c})$ satisfy right hand rule if

$$0 < (\vec{a}\times\vec{b}\cdot\vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \cdot (c_1\vec{i}+c_2\vec{j}+c_3\vec{k}) =$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} c_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} c_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} c_3 =$$

$$= \det\begin{pmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} = \det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \det\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} =$$

$$= dx\wedge dy\wedge dz\,(\vec{a},\vec{b},\vec{c}).$$

Conclusion $dx\wedge dy\wedge dz$ orients $\mathbb{R}^3$.
$(-dx\wedge dy\wedge dz$ gives the opposite orientation$)$.

What about $\mathbb{R}^1$? dx orients it.

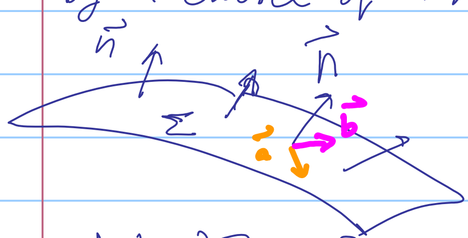
$\boxed{\text{In general } dx_1\wedge dx_2\wedge\ldots dx_n \text{ orients } \mathbb{R}^n.}$

Note If $f(x,y)$ is a positive function, then
$$\big(f(x,y)\,dx\wedge dy\big)(\vec{i},\vec{j}) = f(x,y)\big(dx\wedge dy\,(\vec{i},\vec{j})\big) = f(x,y)\cdot 1 > 0.$$

Therefore, any 2-form μ, which is never 0, orients $\mathbb{R}^2$.
(Such forms are called <u>area forms</u>)

Does this work for surfaces? Yes.

Suppose $\Sigma \subset \mathbb{R}^3$ is a surface, which is oriented by by a choice of a normal $\vec{n}$:



Two vectors $\vec{a}, \vec{b}$, tangent to Σ, Orient Σ positively if
$$dx \wedge dy \wedge dz\,(\vec{n}, \vec{a}, \vec{b}) > 0.$$

<u>Note</u> 1) This is exactly the right hand rule for orienting boundaries of regions in $\mathbb{R}^3$

2) $\Omega_\Sigma(\vec{a}, \vec{b}) := dx \wedge dy \wedge dz(\vec{n}, \vec{a}, \vec{b})$ is a 2-form on Σ

3) Any parameterization $X: D \to \Sigma$ $(D \subseteq \mathbb{R}^2)$ orients Σ: it defines a normal $\vec{N} = \frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v}$.
By (2) above, a choice of a normal gives Σ an area form Ω_Σ.

<u>Conclusions:</u>

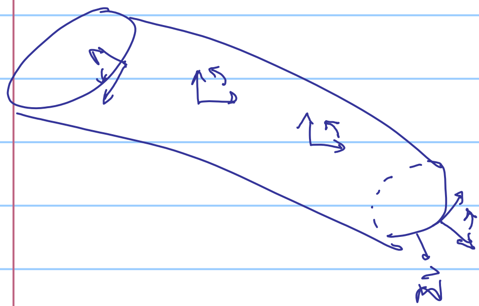
<u>Surfaces</u> are oriented by
ordered pairs of vectors / area forms
Orientations are induced by normals / parameterizations.

Similarly <u>Curves</u> are oriented by
tangent vectors / nowhere zero 1-forms.

<u>k-manifolds</u> are oriented by
k-tuple of vectors / nowhere zero k-forms.
(k-volume forms)

Induced orientations

Let M be a k-manifold with boundary ∂M. Suppose Ω is a k-volume form on M and $\vec{N}$ is the normal on the boundary pointing out of M.

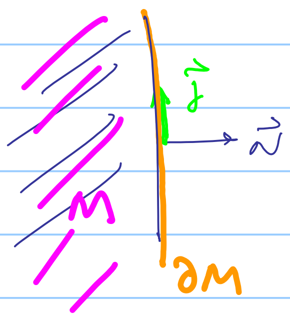


We define a $(k-1)$ volume form $\Omega_{\partial M}$ on ∂M as follows:

$$\Omega_{\partial M}(\vec{V}_1, \dots \vec{V}_{k-1}) := \Omega(\vec{N}, \vec{V}_1, \dots \vec{V}_{k-1}).$$

Example $M = \{(x,y) \mid y \leq 0\}$. $\partial M = \{(x,y) \mid y = 0\}$

$\Omega = dx \wedge dy$ $\quad \vec{N} = \vec{\imath}$



$$\Omega_{\partial M}(\vec{\jmath}) = \Omega(\vec{\imath}, \vec{\jmath}) = dx \wedge dy(\vec{\imath}, \vec{\jmath}) = 1 > 0.$$

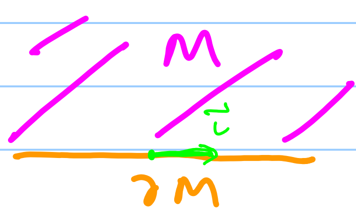
Therefore $\vec{\jmath}$ orients ∂M correctly.

[Note that since $dy(\vec{\jmath}) = 1$,

$\Omega_{\partial M} = dy$]

Example $M = \{(x,y) \mid x \geq 0\}$, $\Omega = dx \wedge dy$

$\vec{N} = -\vec{\jmath}$.



$$\Omega_{\partial M}(\vec{\imath}) = \Omega(-\vec{\jmath}, \vec{\imath}) = dx \wedge dy(-\vec{\jmath}, \vec{\imath}) =$$

$$= \det\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = +1.$$

Therefore $\vec{\imath}$ orients ∂M correctly

[and $\Omega_{\partial M} = dx$].