Lecture 38 12/7/05

Note Title 12/6/2005

Last time:

1) An orientation of a k-manifold M is determined by a k-form Ω which is $\neq 0$ nowhere. The book calls such a form an <u>orientation form.</u> I called it a <u>k-volume form</u>.

2)

Now suppose M is a k-manifold with boundary ∂M. Recall that the orientation of ∂M is defined as follows: Let Ω be the orientation form on M. Choose an outward-pointing normal $\vec{N}$ along ∂M.

The orientation $\Omega_{\partial M}$ is defined by:

(*) $\Omega_{\partial M}(\vec{V}_1, \ldots \vec{V}_{k-1}) = \Omega(\vec{N}, \vec{V}_1, \ldots \vec{V}_{k-1})$.

[More about this shortly].

Now suppose $X: D \to M$ is a parameterization. Then X defines an orientation of M. How do we know if a given orientation Ω of M is <u>compatible</u> with X?

Simple:

$$X^*\Omega = f(u_1 \ldots u_k)\, du_1 \wedge \ldots \wedge du_k.$$

If $f > 0$, it's compatible.

If $f < 0$, it's the opposite.

<u>Example.</u> $M = \{(x,y) \mid x^2 + y^2 = 1\}$, $\Omega = y\,dx - x\,dy$, $X(t) = (\cos t, \sin t)$ a parameterization. Is X compatible with Ω?

$X^*\Omega = \sin t\, d(\cos t) - \cos t\, d(\sin t) = -\sin^2 dt - \cos^2 t\, dt = -dt$

<u>NO</u>.

Remark (**) defines $\Omega_{\partial M}$ by

$$\Omega_{\partial M}(\cdot, \dots \cdot) = \Omega(\vec{N}, \dots).$$

Notation $\iota(\vec{N})\Omega = \Omega(\vec{N}, \dots)$, contraction of $\vec{N}$ and Ω.

Here is a quick way to compute contractions:

$$\iota(\vec{N})\, dx_1 \wedge \dots \wedge dx_k = dx_1(\vec{N})\, dx_2 \wedge \dots \wedge dx_k - dx_2(\vec{N})\, dx_1 \wedge dx_3 \wedge \dots dx_n +$$
$$+ (-1)^{j-1} dx_j(\vec{N})\, dx_1 \wedge \dots \widehat{dx_j} \wedge \dots \wedge dx_n + \dots$$

$\wedge$ = omitted

Example $M = \{(x, y, z) \mid x^2+y^2+z^2 \le 1\}$. $\Omega = dx \wedge dy \wedge dz$

$\partial M = \{(x, y, z) \mid x^2+y^2+z^2 = 1\}$. $\vec{N} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\Omega_{\partial M} = \iota(\vec{N})\, dx \wedge dy \wedge dz = dx(\vec{N})\, dy \wedge dz - dy(\vec{N})\, dx \wedge dz +$$
$$dz(\vec{N})\, dx \wedge dy = x\, dy \wedge dz - y\, dx \wedge dz + z\, dx \wedge dy.$$

$X: \{(u, v) \mid u^2+v^2 < 1\} \to \partial M$, $X(u, v) = (u, v, \sqrt{1-u^2-v^2})$ is a parameterization of a piece (half) of ∂M.

Is it compatible with $\Omega_{\partial M}$?

$$X^* \Omega_{\partial M} = u\, dv \wedge d\sqrt{1-u^2-v^2} - v\, du \wedge d(\sqrt{1-u^2-v^2}) + \sqrt{1-u^2-v^2}\, du \wedge dv$$
$$= u\, dv \wedge \left(\frac{-2u}{2\sqrt{\ }}\, du\right) - v\, du \wedge \left(\frac{-2v}{2\sqrt{\ }}\, dv\right) + \sqrt{\ }\, du \wedge dv =$$
$$= \underbrace{\left(\sqrt{1-u^2-v^2} + \frac{u^2}{\sqrt{1-u^2-v^2}} + \frac{v^2}{\sqrt{1-u^2-v^2}}\right)}_{>0} du \wedge dv$$

yes.

Example

$M = \{(x_1, \dots x_5) \in \mathbb{R}^5 \mid x_5 = x_1 x_2 x_3 x_4,\ 0 \le x_1, x_2, x_3, x_4 \le 1\}$.

Orientation: $\Omega = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$.

Compute $\int_M dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_5$.

Solution Let $D = \{(u_1, \dots u_4) \in \mathbb{R}^4 \mid 0 \le u_1, \dots u_4 \le 1\}$.

$X: D \to M$, $X(u_1, u_2, u_3, u_4) = (u_1, u_2, u_3, u_4, u_1 u_2 u_3 u_4)$

$X^*\Omega = du_1 \wedge du_2 \wedge du_3 \wedge du_4 \Rightarrow X$ is compatible with Ω.

Therefore $\int_M dx_1\wedge dx_2\wedge dx_3\wedge dx_5 = \int_D X^*(dx_1\wedge dx_2\wedge dx_3\wedge dx_5) =$

$$= \int_D du_1\wedge du_2\wedge du_3\wedge d(u_1u_2u_3u_4) = \int_D du_1\wedge du_2\wedge du_3\wedge(u_1u_2u_3\,du_4)$$

$$= \int_0^1\int_0^1\int_0^1\int_0^1 u_1u_2u_3\,du_1du_2du_3du_4 = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot 1 = \frac{1}{8}.$$

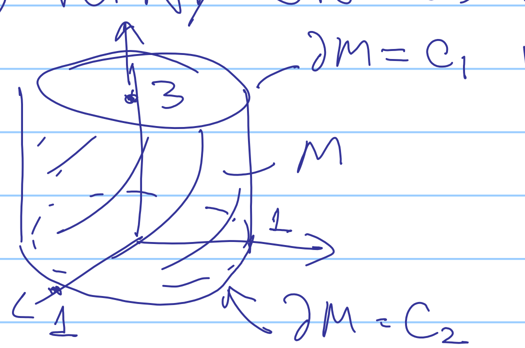
Example Let M be a portion of the cylinder

$$M = \{(x,y,z)\mid x^2+y^2=1,\ 0\le z\le 3\},$$

oriented with the normal $\vec{N} = (x,y,0)$.

a) Use $\vec{N}$ to give M an orientation form Ω

b) Verify Stokes's theorem for $\omega = y\,dx - x\,dy + (x+y+z)dz$



$$\Omega = \iota(\vec{N})\,dx\wedge dy\wedge dz =$$
$$= dx(\vec{N})\,dy\wedge dz - dy(\vec{N})\,dx\wedge dz + dz(\vec{N})\,dx\wedge dy =$$
$$= x\,dy\wedge dz - y\,dx\wedge dz$$

$\partial M = C_1\cup C_2$, $C_1 = \{(x,y,3)\mid x^2+y^2=1\}$,
$C_2 = \{(x,y,0)\mid x^2+y^2=1\}$.

Orientation of C_1: choose $\vec{V} = \vec{k}$ as normal.

$$\Omega_{C_1} = \iota(\vec{V})(x\,dy\wedge dz - y\,dx\wedge dz) =$$
$$= -x\,dz(V)\,dy - y(-dz(\vec{V}))\,dx$$
$$= y\,dx - x\,dy$$

Parameterize C_1 by $X(t) = (\cos t, \sin t, 3)$

$X^*(\Omega_{C_1}) = \sin t\, d(\cos t) - \cos t\, d(\sin t) = (-\sin^2 t - \cos^2 t)dt = (-1)dt$

Wrong orientation. Therefore

$$\int_{C_1}\omega = -\int_{[0,2\pi]} X^*(y\,dx - x\,dy + (x+y+z)dz) =$$

$$= -\int_{[0,2\pi]} \sin t\, d(\cos t) - \cos t\, d(\sin t) + (\cos t + \sin t + 3)\,d(3) =$$

$= -\int_{[0,2\pi]} (-\sin^2 t - \cos^2 t)\, dt = 2\pi$

Parameterize C_2 by $X(t) = (\cos t, \sin t, 0)$. Is this correct? Does it give the right orientation?

$\Omega_{C_2} = \iota(-\vec{k})(x\, dy \wedge dz - y\, dx \wedge dz) = \left(-dz(-\vec{k})\right) x\, dy$
$\left(-dz(-\vec{k})\right)(-y\, dx) = x\, dy - y\, dx.$

$X^*(x\, dy - y\, dx) = \cos t\, d\sin t - \sin t\, d\cos t = dt$ ✓ yes

$\Rightarrow \int_{C_2} \omega = \int_{[0,2\pi]} X^*\omega = \int_{[0,2\pi]} X^*(y\, dx - x\, dy + (x+y+z)\, dz) = \int_{[0,2\pi]} \sin t\, d\cos t -$
$- \cos t\, d(\sin t) + (\cos t + \sin t + 0)\, d\,0) = \int_{[0,2\pi]} (-dt) = -2\pi.$

$\therefore \int_{\partial M} \omega = 0.$

Now parameterize M by
$X : [0,2\pi] \times [0,3] \to M$
$X(u_1, u_2) = (\cos u_1, \sin u_1, u_2).$

$X^*\Omega = \cos u_1\, d(\sin u_1) \wedge du_2 - \sin u_1\, d\cos u_1 \wedge du_2 =$
$= (\cos^2 u_1 + \sin^2 u_1)\, du_1 \wedge du_2 = du_1 \wedge du_2.$ ✓ good.

$d\omega = d(y\, dx - x\, dy + (x+y+z)\, dz) = dy \wedge dx - dx \wedge dy + dx \wedge dz + dy \wedge dz$

$\Rightarrow \int_M d\omega = \int_{[0,2\pi]\times[0,3]} X^*(2\, dy \wedge dx + dx \wedge dz + dy \wedge dz) =$

$= \int_{[0,2\pi]\times[0,3]} (2\, d(\sin u_1) \wedge d\cos u_1 + d\cos u_1 \wedge du_2 + d\sin u_1 \wedge du_2)$

$= \int_{[0,2\pi]\times[0,3]} (-\sin u_1 + \cos u_1)\, du_1 \wedge du_2 = \int_0^{2\pi}\int_0^3 (-\sin u_1 + \cos u_1)\, du_2\, du_1 =$

$= 3 \int_0^{2\pi} (-\sin u_1 + \cos u_1)\, du_1 = 0.$