

**Set 2 of problem bank questions, Math 425, Prof. Eugene Lerman
(Do not turn in. May show up on a midterm.)**

The goal of this set of problems is understand continuity for linear maps between normed vector spaces. It will turn out that linear maps between finite dimensional vector spaces are always continuous but for infinite dimensional vector spaces this may not be true.

Let V and W be two vector spaces with norms $\|\cdot\|_V$ and $\|\cdot\|_W$. A linear map $T : V \rightarrow W$ is **bounded** if there is $C > 0$ so that

$$\|Tv\|_W \leq C\|v\|_V$$

for all $v \in V$.

1 Prove that a linear map $T : V \rightarrow W$ is bounded if and only if it is continuous at 0.

2 Prove that a linear map $T : V \rightarrow W$ is continuous at 0 if and only if it is continuous at all points of V .

3 Prove that if $V = \mathbb{R}^n$, $W = \mathbb{R}^m$ and the norms are Euclidean norms $\|\cdot\|_2$ than any linear map $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is continuous/bounded. Hint: recall that $\|Av\|_2 \leq \|A\|_2\|v\|_2$ for any matrix A of the appropriate size.

4 Let $Q \subset \mathbb{R}^n$ be a rectangle. Check that the map from the space of bounded functions $B(Q)$ to $[0, \infty)$, $f \mapsto \sup_{x \in Q} |f(x)|$ is a norm. It is called the sup norm or the infinity norm and is often denoted by $\|\cdot\|_\infty$.

5 Since the space of Riemann integrable functions $R^1(Q)$ is a subspace of $B(Q)$ by definition, it also gets the norm $\|\cdot\|_\infty$. Prove that integration

$$\int_Q : R^1(Q) \rightarrow \mathbb{R}$$

is a bounded map, hence continuous. (The norm on \mathbb{R} is the absolute value). Hint: a problem on Homework #4.

6 The point of this exercise is to construct an example of a linear map between two normed vector spaces that is not bounded. Let V be the vector space $C^1([-1, 1])$ with the sup norm and $W = C^0([-1, 1])$, also with the sup norm. Show that the derivative $f(x) \mapsto f'(x)$ is not bounded. Hint: we will prove that there is a function $\phi \in C^1([-1, 1])$ with $\phi(x) \geq 0$ and $\phi(x) = 0$ for $|x| \geq 1/2$. Now consider the sequence $\phi_n(x) := \phi(x/n)$, $n = 1, 2, \dots$