Hilbert schemes of points

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Outline

Hilbert scheme of points on $\mathbb{C}^2$

Hilbert scheme of points on a Deligne-Mumford stack

Hilbert scheme and $q$, $t$-Catalan numbers
Hilbert scheme of points on $\mathbb{C}^2$

$\text{Hilb}^n(\mathbb{C}^2)$ is a scheme that parameterizes 0-dimensional subschemes $Z \subset \mathbb{C}^2$ satisfying $\dim \mathcal{O}_Z = n$. 
Hilbert scheme of points on $\mathbb{C}^2$

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Example

- $n = 1$. $\text{Hilb}^1(\mathbb{C}^2) = \mathbb{C}^2$.
- $n = 2$. $\text{Hilb}^2(\mathbb{C}^2) = (\text{Bl}_\mathcal{I} \mathbb{C}^4)/S_2$, where
  \[ \mathcal{I} = (x_1 - x_2, y_1 - y_2). \]
- For general $n$. $\text{Hilb}^n(\mathbb{C}^2) = (\text{Bl}_\mathcal{I} \mathbb{C}^{2n})/S_n$, where
  \[ \mathcal{I} = \cap_{1 \leq i < j \leq n}(x_i - x_j, y_i - y_j). \]
Properties of $\text{Hilb}^n(\mathbb{C}^2)$:

- $\text{Hilb}^n(\mathbb{C}^2)$ is smooth and connected.
- $\text{Hilb}^n(\mathbb{C}^2)$ has a cellular decomposition.
- There is a Hilbert-Symm morphism $\text{Hilb}^n(\mathbb{C}^2) \to \text{Sym}^n(\mathbb{C}^2)$.
- $\text{Hilb}^n(\mathbb{C}^2)$ is holomorphic symplectic, hence gives a crepant resolution of $\text{Sym}^n(\mathbb{C}^2)$. 
Hilbert scheme of points on a Deligne-Mumford stack (a project suggested by J. Starr)

Assume $k = \bar{k}$,

$\mathcal{X}$ is a tame DM stack / $k$ and is a global quotient, the coarse moduli space $X$ is (quasi-)projective.
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$\mathcal{X}$ is a tame DM stack $\rightarrow k$ and is a global quotient, the coarse moduli space $X$ is (quasi-)projective.

Definition

$\text{Hilb}^n(\mathcal{X})$ is the (quasi-)projective scheme that represents the functor

$$
\begin{align*}
T \rightarrow \left\{ \mathcal{C} \subset \mathcal{X} \times T \right\} & \quad \left( k\text{-Schemes} \right) \rightarrow \left( \text{Sets} \right) \\
\text{where } \mathcal{C} \text{ is a closed substack, finitely presented, flat and proper over } T, \\
\text{satisfy the Hilbert polynomial condition (*)} & \\
\end{align*}
$$

Recall: \( \forall \) coherent \( \mathcal{O}_X \)-module \( F \), define

\[
\chi(X, F) = \sum_{i=0}^{\infty} (-1)^i \dim_k H^i_{\acute{e}t}(X, F).
\]

The Hilbert polynomial of \( F \), \( P_F : K^0(X) \to \mathbb{Z} \), is defined as

\[
[\mathcal{E}] \to \chi(X, \mathcal{E} \otimes_{\mathcal{O}_X} F), \quad \forall \text{ locally free sheaf } \mathcal{E}.
\]

Condition (*): \( P_{\mathcal{O}_t}(\mathcal{E}) = n \ \text{rank}\mathcal{E} \quad \forall t \in T. \)
Theorem

Let $\mathcal{X}$ be a smooth 2-dim tame DM stack with (quasi-)projective coarse moduli space $X$. Then $\text{Hilb}^n(\mathcal{X})$ is smooth and (quasi-)projective for all $n \in \mathbb{N}$. 
Theorem

Let $\mathcal{X}$ be a smooth 2-dim tame DM stack with (quasi-)projective coarse moduli space $X$.

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Idea: $\exists$ étale covering $\{X_i \to X\}$, scheme $U_i$ with $G_i$-action,

$\begin{array}{ccc}
[U_i/G_i] & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow \\
X_i & \text{étale} & X
\end{array}$

Let $W =$ the maximal open subscheme of $\text{Hilb}^n([U_i/G_i])$ where the rational map $\text{Hilb}^n([U_i/G_i]) \dashrightarrow \text{Hilb}^n(\mathcal{X})$ is defined.

Show that $W \to \text{Hilb}^n(\mathcal{X})$ is étale.

Then show that $\text{Hilb}^n([U_i/G_i])$ is smooth. □
Proposition

Let $\mathcal{X}$ be a tame DM stack and is a global quotient. Suppose its coarse moduli space $X$ is a quasi-projective scheme. Then there exists a morphism

$$\text{Hilb}^n(\mathcal{X}) \to \text{Sym}^n(X)$$

taking a zero-dimensional substack to the underlying set of points in $X$ over which the substack is supported.

Remark: Neeman showed that $\text{Sym}^n\mathbb{P}^m \to \text{Chow}_{0,n}\mathbb{P}^m$ is not an isomorphism if $\text{char } k = p > 0$ and $n, m \geq p + 1$. 
Theorem

Let \( \mathcal{X} \) be a smooth 2-dim tame DM stack with a connected quasi-projective coarse moduli space \( X \). Assume \( \mathcal{X} \) has only isolated stacky points and each isotropy group is

1. abelian, or,
2. a subgroup of \( SL(2, \mathbb{C}) \) (for \( k = \mathbb{C} \)).

Then the quasi-projective scheme \( \text{Hilb}^n \mathcal{X} \) is connected.
Theorem

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Idea: Consider $\pi : \text{Hilb}^n(\mathcal{X}) \rightarrow \text{Sym}^n(X)$. Since $\text{Sym}^n(X)$ is connected, it suffices to show each fiber of $\pi$ is connected. Each fiber is isomorphic to a fiber of

$$\text{Hilb}^n([\mathbb{A}^2/G]) \rightarrow \text{Sym}^n(\mathbb{A}^2/G).$$

By Zariski’s main theorem, it suffices to show that $\text{Hilb}^n([\mathbb{A}^2/G])$ is connected, which is known under condition (1) or (2). $\square$
Example

For $\mathcal{X} = \mathbb{A}^2 / G$, $\text{Hilb}^n(\mathcal{X})$ are Hilbert schemes of regular $G$-orbits.

- $G = \text{abelian group}$: $\text{Hilb}^n(\mathcal{X})$ is a multigraded Hilbert scheme.
- $n = 1$: $\text{Hilb}^1(\mathcal{X}) = G$-Hilbert schemes.
- $G \subset SL_2(\mathbb{C})$: $\text{Hilb}^n(\mathcal{X})$ is a quiver variety.
Proposition

Suppose \((a, m) = 1, 1 \leq a \leq m - 1\).

\(\mu_m\) act on \(\mathbb{C}^2\) as \(\omega \cdot (x, y) = (\omega x, \omega^a y)\) where \(\omega = e^{2\pi i / m}\).

Let \(\tilde{X}\) be the minimal resolution of \(\mathbb{C}^2/\mu_m\).

Then the natural birational map

\[
\text{Hilb}^n(\tilde{X}) \dashrightarrow \text{Hilb}^n([\mathbb{C}^2/\mu_m])
\]

is not a morphism for \(n \geq 2\).
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Remark: Take \(a = m - 1\). Then \(\text{Hilb}^n(\tilde{X})\) and \(\text{Hilb}^n([\mathbb{C}^2/\mu_m])\) give different crepant resolutions of \(\text{Sym}^n(\mathbb{C}^2/\mu_m)\) for \(n \geq 2\).
Betti numbers and cellular decomposition of $\text{Hilb}^n([\mathbb{A}^2/\mu_m])$

**Theorem**

Let $n, a, b, m \in \mathbb{N}$ such that $\gcd(a, b, m) = 1$. Let $\mu_m$ acting on $\mathbb{A}^2$ as $\omega(x, y) = (\omega^ax, \omega^by)$ where $\omega^m = 1$. Then $\text{Hilb}^n([\mathbb{A}^2/\mu_m])$ is a smooth irreducible quasi-projective scheme with a cellular decomposition, and this decomposition is described in terms of certain combinatoric data.
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**Theorem**

Let $n, a, b, m \in \mathbb{N}$ such that $\text{gcd}(a, b, m) = 1$. Let $\mu_m$ acting on $\mathbb{A}^2$ as $\omega(x, y) = (\omega^a x, \omega^b y)$ where $\omega^m = 1$. Then $\text{Hilb}^n(\mathbb{A}^2/\mu_m)$ is a smooth irreducible quasi-projective scheme with a cellular decomposition, and this decomposition is described in terms of certain combinatoric data.

**Example**

For $n = 2, a = 1, b = 1, m = 2$, there are five admissible Young diagrams.

\[
\begin{array}{c}
\begin{array}{c}
0 & 1 & 0 & 1 \\
0 & 1 & & 0
\end{array}
\end{array}
\]
\[d(D)=0\]

\[
\begin{array}{c}
\begin{array}{c}
1 & 0 \\
1 & 0
\end{array}
\end{array}
\]
\[d(D)=1\]

\[
\begin{array}{c}
\begin{array}{c}
1 & 0 & 1 & 0 \\
0 & 1 & & 0
\end{array}
\end{array}
\]
\[d(D)=1\]

\[
\begin{array}{c}
\begin{array}{c}
1 \\
0 & 1
\end{array}
\end{array}
\]
\[d(D)=2\]

\[
\begin{array}{c}
\begin{array}{c}
1 & 0 \\
1 & 0
\end{array}
\end{array}
\]
\[d(D)=2\]

$\text{Hilb}^2(\mathbb{A}^2/\mu_2) \cong \mathbb{A}^4 + 2\mathbb{A}^3 + 2\mathbb{A}^2$, \quad $\text{Hilb}^2_0(\mathbb{A}^2/\mu_2) \cong \mathbb{A}^0 + 2\mathbb{A}^1 + 2\mathbb{A}^2$. 
Summary:

- Define $\text{Hilb}^n(\mathcal{X})$ for a tame DM-stack.
- Let $\mathcal{X}$ be a smooth tame DM-stack of dim 2,
  - $\text{Hilb}^n(\mathcal{X})$ is smooth.
  - give sufficient conditions for $\text{Hilb}^n(\mathcal{X})$ to be connected.
  - for $\mathcal{X} = \mathbb{A}^2/\mu_m$, $\text{Hilb}^n(\mathcal{X})$ has a cellular decomposition.
Summary:

- Define $\text{Hilb}^n(X)$ for a tame DM-stack.
- Let $X$ be a smooth tame DM-stack of dim 2,
  - $\text{Hilb}^n(X)$ is smooth.
  - give sufficient conditions for $\text{Hilb}^n(X)$ to be connected.
  - for $X = [\mathbb{A}^2/\mu_m]$, $\text{Hilb}^n(X)$ has a cellular decomposition.

Ongoing research:

- The global geometry of $\text{Hilb}^n(X)$ for a toric DM stack $X$.
- Connectedness of $\text{Hilb}^n(X)$ for general $X$. 
Hilbert scheme and \( q, t \)-Catalan numbers (joint with Kyungyong Lee)

Consider the ideal \( I = \cap_{1 \leq i < j \leq n} (x_i - x_j, y_i - y_j) \) of \( \mathbb{C}[x, y] := \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \).
Define \( M = I / (x, y)I \).

**Problem (Haiman)**

*Find an explicit basis of the bi-graded vector space \( M \).*
Hilbert scheme and $q, t$-Catalan numbers (joint with Kyungyong Lee)

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Problem (Haiman)

*Find an explicit basis of the bi-graded vector space $M$.*

Theorem (Haiman)

1. $\dim M = \frac{1}{n+1} \binom{2n}{n}$.
2. $q, t$-Catalan number $C_n(q, t) = \sum_{d_1, d_2} t^{d_1} q^{d_2} \dim M_{d_1, d_2}$.
3. Let $H^n_0$ be the zero fiber of $\text{Hilb}^n(\mathbb{C}^2) \to \text{Sym}^n(\mathbb{C}^2)$. Then
   
   $C_n(q, t) = \sum_{i=0}^{n-1} (-1)^i \text{tr}_{H^i(H^n_0, O(1))}(q, t)$. 

Table of $q, t$-catalan number for $n = 7$.

The coefficient of $q^{d_1} t^{d_2}$ is $p(k)$ for $k = n(n - 1)/2 - d_1 - d_2$, $d_1, d_2 \geq k$.
Theorem

Let $d_1, d_2$ be non-negative integers s.t. $d_1 + d_2 \leq \binom{n}{2}$. Define $k = \binom{n}{2} - d_1 - d_2$ and $\delta = \min(d_1, d_2)$. Then

$$\dim M_{d_1, d_2} \leq p(\delta, k),$$

and the equality holds iff

- $k \leq n - 3$, or
- $k = n - 2$ and $\delta = 1$, or
- $\delta = 0$.

In case the equality holds, there is an explicit construction of a basis of $M_{d_1, d_2}$. 
Idea to prove \( \dim M_{d_1,d_2} \leq p(\delta, k) \)

For any \( n \)-point set \( D = \{ (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n) \} \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \)

Define \( \Delta(D) = \det[x_i^{\alpha_j} y_i^{\beta_j}]_{i,j} \), \( \text{bideg}(D) := (\sum \alpha_j, \sum \beta_j) \).

Then \( \{ \Delta(D) \}_{\text{bideg}(D)=(d_1,d_2)} \) generates \( M_{d_1,d_2} \).
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Example

3-point sets of bidegree (2, 1):

\[
D = \begin{array}{ccc}
1 & x_1^2 & y_1 \\
1 & x_2^2 & y_2 \\
1 & x_3^2 & y_3
\end{array} \quad \Delta(D) = \begin{vmatrix}
1 & x_1^2 & y_1 \\
1 & x_2^2 & y_2 \\
1 & x_3^2 & y_3
\end{vmatrix} \quad D' = \begin{array}{ccc}
1 & x_1 & x_1 y_1 \\
1 & x_2 & x_2 y_2 \\
1 & x_3 & x_3 y_3
\end{array} \quad \Delta(D') = \begin{vmatrix}
1 & x_1 & x_1 y_1 \\
1 & x_2 & x_2 y_2 \\
1 & x_3 & x_3 y_3
\end{vmatrix}
\]

Then \( \Delta(D), \Delta(D') \) generate \( M_{2,1}. \)

But such generators are redundant in general.
For a bidegree \((d_1, d_2)\) satisfying
\[
k := \binom{n}{2} - d_1 - d_2 << n,
\]
there are unique integers \(a_\mu\) such that
\[
\Delta(D) = \sum a_\mu \Delta(F_\mu) \text{ in } M
\]
where \(F_\mu = \) is an \(n\)-point set of bidegree \((d_1, d_2)\) and is of the form

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(in the example, the partition type is \(\mu = (2, 1, 5)\)).

In other words, \(\{\Delta(F_\mu)\}\) form a basis, for \(\mu\) runs through partition of \(k\) into at most \(\delta = \min(d_1, d_2)\) parts.

Therefore
\[
\dim M_{d_1, d_2} \leq p(\delta, k).
\]
Idea to prove \( \dim M_{d_1,d_2} \geq p(\delta, k) \) for \( k \leq n - 3 \).

For each \( D \), by adding sufficient many points, we get \( \tilde{D} \), such that \( \Delta(\tilde{D}) \) can be written uniquely as

\[
\Delta(D) = \sum a_\mu \Delta(F_\mu) \text{ in } M.
\]

Define

\[
\varphi(\Delta(D)) := \sum a_\mu \rho_\mu,
\]

where \( \rho_\mu = \rho_{\mu_1} \rho_{\mu_2} \cdots \rho_{\mu_\ell} \) for \( \mu = (\mu_1, \mu_2, \ldots, \mu_\ell) \).
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$$\Delta(D) = \sum a_\mu \Delta(F_\mu) \text{ in } M.$$

Define

$$\varphi(\Delta(D)) := \sum a_\mu \rho_\mu,$$

where $\rho_\mu = \rho_{\mu_1} \rho_{\mu_2} \cdots \rho_{\mu_\ell}$ for $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$.

Define $\text{weight}(\rho_i) = i$. For $w \in \mathbb{Z}$, $f \in \mathbb{C}[[\rho_1, \rho_2, \ldots ]]$, denote $\{f\}_w = \text{weight-}w$ part of $f$.

Lemma

$$\varphi(\Delta(D)) = (-1)^k \det \left[ \left\{ (1 + \rho_1 + \rho_2 + \cdots)^{\beta_i} \right\}_{j-1-\alpha_i-\beta_i}^{i,j} \right]$$
Proposition

φ induces a linear map \( \bar{\varphi} : M \rightarrow \mathbb{C}[\rho_1, \rho_2, \ldots] \).
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\( \varphi \) induces a linear map \( \bar{\varphi} : M \rightarrow \mathbb{C}[\rho_1, \rho_2, \ldots] \).

For each partition \( \mu \in \Pi(\delta, k) \), we explicitly construct \( D_\mu \), s.t.

\[
LM \varphi(\Delta(D_\mu)) = \rho_\mu.
\]

\( \Rightarrow \) \( \{\bar{\varphi}(\Delta(D_\mu))\} \) are linearly independent

\( \Rightarrow \) \( \{\Delta(D_\mu)\} \) are linearly independent (since \( \bar{\varphi} \) is well-defined).

\( \Rightarrow \) \( \dim M_{d_1,d_2} \geq p(\delta, k) \) for \( k \leq n - 3 \).  

\( \square \)
Summary:

- For $I = \cap(x_i - x_j, y_i - y_j)$, $M = I/(x, y)I$ arises in the study of $\text{Hilb}^n(\mathbb{C}^2)$. Its Hilbert series gives the $q, t$-Catalan number.

- $\dim M_{d_1, d_2} \leq p(\delta, k)$, where $k = \binom{n}{2} - d_1 - d_2$, $\delta = \min(d_1, d_2)$.

- For $k \leq n - 3$, above “=” holds, we find an explicit basis for $M_{d_1, d_2}$. 
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- For $I = \cap(x_i - x_j, y_i - y_j)$, $M = I/(x, y)I$ arises in the study of $\text{Hilb}^n(\mathbb{C}^2)$. Its Hilbert series gives the $q, t$-Catalan number.

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- For $k \leq n - 3$, above “=” holds, we find an explicit basis for $M_{d_1, d_2}$.

Ongoing research:

- Conjectural basis for $M_{d_1, d_2}$.

- Extend our method to the study of $I = \cap(x_i - x_j, y_i - y_j, z_i - z_j)$ and $\text{Hilb}^n(\mathbb{C}^3)$. 

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Hilbert schemes of points