

## 1. HOMEWORK THAT WAS DUE AUGUST 29

The homework system was down the evening of August 28. Almost everybody did the first homework before that, and got full credit. To be fair, I will give everybody full credit for the first homework.

## 2. HOMEWORK DUE THURSDAY, AUGUST 31 AT 9 A.M.

Section 1.3: #4, 8, 10, 12, 14, and 16.

Section 1.4: #14, 16, 22, and 24.

This second homework with graphs is now online as HW2NewAug31. You will go over this homework this week, but you can still work it until noon on Sunday. Some people guessed and got good scores, but to get the benefit of the homework, I am asking that it be done with the graphs. If you somehow already did this homework with the graphs, let me know and I will give you the score that was recorded before the system went down. I hope for smooth sailing from now on.

## 3. HOMEWORK DUE TUESDAY, SEPTEMBER 5 AT 9 A.M.

Section 2.1: #4, 8, 18, 22, 26.

Section 2.2: #4, 14, 22, 30, 32, 34, 36. Give exact answers.

## 4. WRITTEN PROBLEM FOR NEXT WEEK

Show that  $\lim_{x \rightarrow 2} x^2 + 5x - 7 = 7$  **using the facts** that  $\lim_{h \rightarrow 0} h^2 = 0$  and  $\lim_{h \rightarrow 0} 9h = 0$ .

You must use these facts in your demonstration to get credit for the problem.

Hint: You need to show that  $\lim_{h \rightarrow 0} [(2+h)^2 + 5(2+h) - 14] = 0$ .

## 5. NUMBERS

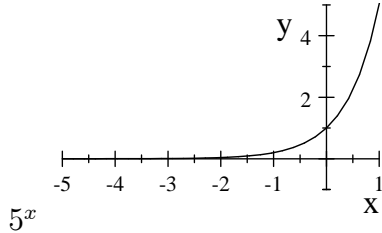
The **natural numbers** are the whole numbers 1, 2, 3, etc. The **integers** are the natural numbers, the negatives of the natural numbers, and 0. The **rational numbers** are the numbers that can be expressed as ratios of integers, that is, fractions where the numerator and denominator are integers and the denominator is not 0. Two fractions  $n/m$  and  $k/j$  represent the same rational number if  $nj = km$ . There are real numbers that are not rational numbers. An example is  $\sqrt{2}$ .

## 6. EXPONENTIAL AND LOGARITHMIC FUNCTIONS

**Discussion in class postponed.** Along with the trigonometric functions, sin, cos, tan, etc., we will look at functions of the form  $a^x$  where  $a$  is a constant. We will assume that  $a > 1$  so that we get a function for which the  $y$ -values increase as the  $x$ -values increase. We know that  $a^0 = 1$ , that  $a^1 = a$ . For an integer  $n > 1$ ,  $a^n = a \cdot a \cdots a$  where there are  $n$  factors. We also know that for a positive rational number,  $\frac{n}{m}$ ,  $a^{\frac{n}{m}}$  is the  $m^{\text{th}}$ -root of  $a^n$ , and  $a^{-\frac{n}{m}} = 1/a^{\frac{n}{m}}$ . This gives us a function defined on the rational

numbers. We then can define  $a^x$  for all numbers  $x$  by filling in the holes in the graph (that is, by taking “limits” over the rationals.)

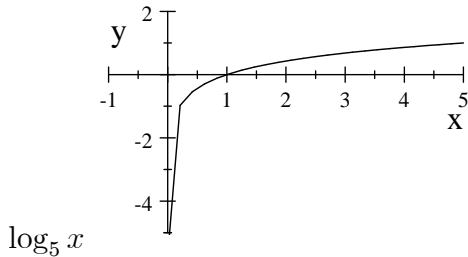
The graph of  $y = a^x$  is that of an everywhere positive function of  $x$  for which the  $y$ -value increases as the  $x$ -value increases. The limit as  $x \rightarrow -\infty$  is 0, and the limit as  $x \rightarrow +\infty$  is  $+\infty$ . The graph is concave up. That is, the graph is always below a straight line segment joining two points on the graph.



Common values for  $a$  are 2 and 10. For example,  $2^3 = 8$ ,  $10^3 = 1000$ . We will later study the number  $e = 2.718281828459\dots$ . This is the base for “natural logs”.

Corresponding to the function  $y = a^x$  is the inverse function  $y = \log_a x$ . That is,  $y$  is the power it is necessary to raise  $a$  to in order to obtain  $x$ . For example,  $\log_2 8 = 3$ , and  $\log_{10} 1000 = 3$ .

Since  $a^x$  is always positive, the graph of  $y = \log_a x$  is only defined for  $x > 0$ . The graph is increasing. That is, the  $y$ -values increase as the  $x$ -values increase. The graph is concave down (i.e., the line segment joining two points on the graph lies below the graph), and it crosses the  $x$ -axis at  $x = 1$  and takes the value 1 at  $a$ . The limit as  $x \rightarrow 0$  is  $-\infty$ , and the limit as  $x \rightarrow +\infty$  is  $+\infty$ .



## 7. RATES OF CHANGE

For a while, now we will in one way or another be talking about rates of change. Velocity, for example, is the rate of change of distance traveled with respect to the time it takes to do the traveling.

If you are building a levee to prevent a river from overflowing and it is raining hard, you would like to know the water level  $w$  as a function of time  $t$ . You would also like to know the rate of change of  $w$  at any time. The most important information you need is what the maximum value of  $w$  will be. The maximum value of  $w$  will be realized when the water stops rising and starts to fall; that is, when the rate of

change of  $w$  switches from being positive to being negative. At that time, the rate of change of  $w$  will be 0.

If we are given a graph of a function  $y = f(x)$ , and a point  $P = (x_0, y_0)$  on that graph, we would like to talk about the best straight line approximation to the graph at the point  $P$ . One idea is that the line approximation would be the path that a car traveling along the graph would take if at the point  $P$ , all forces became zero on the car. For example, if you twirl a ball on a string around your head and then let go, in the absence of gravity, the ball would follow such a straight line path. We will call the best line approximation to the graph at  $P$  the tangent line to the graph at  $P$ . This is because we think of the line as just touching the graph at  $P$ . What does this have to do with rates of change?

In general, the problem can be stated as follows: We are given a function  $y = f(x)$  and we would like to say what we mean by the rate of change of the function at a point  $x$ . We know the average rate of change over the interval  $[x, x + h]$  where  $h > 0$ . This is  $(f(x + h) - f(x))/h$ , and we can visualize this as the slope of the line going through the points  $(x, f(x))$  and  $(x + h, f(x + h))$  on the graph of  $f$ . A similar calculation can be made for negative values of  $h$ . What we would like is for all of these ratios, both for  $h > 0$  and for  $h < 0$  to have a limit as  $h$  goes to 0. If this is true, we will write  $f'(x)$  for that value. If the value  $f'(x)$  exists, it makes sense to think of it as the slope of the tangent line to the graph of  $f$  at the point  $(x, f(x))$ . We will call  $f'(x)$  the **derivative** of the function  $f$  at the point  $x$ . For now, your book calls it the **slope-predictor** for  $f$  at  $x$ . Wherever the derivative exists, it gives us a new function  $f'$  of the variable  $x$ . If we set  $\Delta x = h$ , then  $f'(x)$  is the limiting value, if it exists, of the ratio  $\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{\Delta y}{\Delta x}$ . Note that we do not let  $\Delta x = 0$ .

We will **define the tangent line** to the graph of  $f$  at the point  $(x_0, f(x_0))$  as the line through this point with slope  $f'(x_0)$ . That is, the equation of the tangent line at this point is given by

$$f'(x_0) = \frac{y_{line} - f(x_0)}{x_{line} - x_0}$$

or

$$y_{line} = f'(x_0)(x_{line} - x_0) + f(x_0).$$

I have written  $y_{line}$  and  $x_{line}$  because you must distinguish between the variables for the original function and the variables for the line. When you are writing down the equation for a tangent line at a point, don't call the point  $(x, y)$  or even  $(x, f(x))$ , since if you do you will get confused with the variables for points on the line. Also, don't write the slope as a function of  $x$  if you are working at a single point. It must be clear that the slope at that point is a fixed number.

Here is an example. If  $y = f(x) = x^2$ , then at any point  $(x, y)$ , the slope of the tangent line is

$$\begin{aligned} \text{limit as } \Delta x \text{ goes to } 0 \left[ \frac{(x + \Delta x)^2 - x^2}{\Delta x} \right] &= \\ \text{limit as } \Delta x \text{ goes to } 0 \left[ \frac{2x \cdot \Delta x + (\Delta x)^2}{\Delta x} \right] &= \\ \text{limit as } \Delta x \text{ goes to } 0 [2x + \Delta x] &= 2x. \end{aligned}$$

This means that at  $x = 3$ ,  $y = 9$ , the slope of the tangent line is 6, and the equation of the tangent line is  $y - 9 = 6(x - 3)$ . If we write the slope at this point as  $2x$ , then we have the equation  $y - 9 = 2x(x - 3) = 2x^2 - 6x$ . This is the equation of a parabola, not a line.

To begin with, we will consider rates of changes and corresponding tangent lines for lines or parabolas. Consider the function  $y = f(x) = ax^2 + bx + c$ . If  $a = 0$ , this is the equation of a straight line. We will let  $\Delta x = h$  to simplify notation. At any point  $x_0$  and for any  $\Delta x = h \neq 0$ ,

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + h) - f(x_0)}{h} = \frac{[a(x_0 + h)^2 + b(x_0 + h) + c] - [a(x_0)^2 + bx_0 + c]}{h}.$$

That is,

$$\frac{\Delta y}{\Delta x} = \frac{2ax_0h + ah^2 + bh}{h} = 2ax_0 + ah + b.$$

Since  $a$  is a fixed constant, as  $h$  gets smaller and smaller,  $ah$ , gets smaller and smaller. On the other hand,  $2ax_0 + b$  does not change as  $h$  changes. Therefore, as we will clarify later, as  $\Delta x = h$  goes to 0, the ratio  $\Delta y/\Delta x$  has limiting value  $2ax_0 + b$ . Therefore we have the following rule for the derivative or “slope-predictor” at  $x$ :

$$\text{If } f(x) = ax^2 + bx + c \quad \text{then} \quad f'(x) = 2ax + b.$$

You need to remember this fact. Notice that the slope of the tangent line does not change as we change the constant  $c$ .

**EXAMPLE:** Let  $f(x) = 3x^2 + 2x + 1$ . Then  $f'(x) = 6x + 2$  at each  $x$ . The tangent line to the parabola  $y = 3x^2 + 2x + 1$  at the point  $(1, 6)$  has slope 8. That is, if we now let  $x$  and  $y$  denote the variables for the tangent line,  $\frac{y-6}{x-1} = 8$ , i.e.,  $y = 8x - 8 + 6 = 8x - 2$  is the equation for the tangent line to the graph of  $f$  at  $(1, 6)$ . One check is to make sure the tangent line we describe has the right slope 8 and goes through the point  $(1, 6)$ .

**NOTE:** The **normal line** to a graph at a point is a line through the point perpendicular to the tangent line. That is, since  $f'(x_0)$  is the slope of the tangent

line at  $(x_0, f(x_0))$ , if  $f'(x_0) \neq 0$ , the slope of the normal line at the same point is  $-1/f'(x_0)$ . That is, the equation for the normal line is

$$y = \frac{-1}{f'(x_0)} \cdot (x - x_0) + f(x_0).$$

**EXAMPLE:** If at  $(14, 5)$  the tangent line has slope 7, then the equation for the normal line at  $(14, 5)$  is  $y = (-1/7)(x - 14) + 5 = -x/7 + 7$ .

### 8. LIMIT 0 AT 0

We have seen that to get the derivative, i.e., slope predictor, at a point, we need to take the limiting value as  $h = \Delta x$  goes to 0. Almost all the limits we consider in this course will be of this kind. Moreover, other limits can be simplified by reducing them to limits of this kind. Therefore, I will start with this simple case of a general limit. It is not a new idea, it is the simplest case of the general idea. Every other limit can be reduced to this special, simple case. So what do we mean when we say that the output  $y$  has limit 0 as the input goes to 0? I will give you the formal definition. You don't have to remember the definition or the proofs of the 5 simple rules that follow from it, but you do have to remember those 5 rules.

**Definition 1.** *Let  $f$  be a function of  $h$  defined on some open interval about 0 except perhaps at 0. Then the limit of  $f(h)$  at 0 is 0 if for any  $\varepsilon > 0$ , there is a  $\delta > 0$  such that for each  $h \neq 0$  with  $-\delta < h < \delta$ , we have  $-\varepsilon < f(h) < \varepsilon$ . We write  $\lim_{h \rightarrow 0} f(h) = 0$ .*

The idea is that no matter how small you require the output  $f(h)$  to be, I can find an interval about 0 such that for every number  $h$  except perhaps 0 in that interval,  $f(h)$  meets your requirement. We do not care what happens when  $h = 0$ . That is, you can make the output as small as you like (in absolute value) by making the input sufficiently small.

**EXAMPLE:** Consider the function  $3 \cdot h$ . If we want to make this smaller than  $1/100$ , we just make the input smaller than  $1/300$ . In general, to make it smaller than some positive  $\varepsilon$ , all we have to do is restrict  $h$  to the interval  $(-\varepsilon/3, \varepsilon/3)$ , i.e.,  $\delta = \varepsilon/3$ .

Here are some examples of functions with limit 0 at 0.

i) The function  $f(h) = h$  and the function  $ab(h) = |h|$ .

All we have to do is set  $\delta = \varepsilon$ .

ii) For any constant  $c$ ,  $E(h) = c \cdot h$ .

If  $c = 0$ , all values of  $\delta$  will work. Otherwise we set  $\delta = \varepsilon/|c|$ .

- iii) If  $g$  is a function of  $h$  and on some open interval about 0 we have  $|g(h)| \leq c$  for some constant  $c$ , then  $h \cdot g(h)$  has limit 0 at 0. This reduces to Example ii.

The rules for general limits reduce to the rules for functions with limit 0 at 0. Therefore, we start with those simple rules.

### 9. RULES FOR FUNCTIONS WITH LIMIT 0 AT 0.

I give the proof that each rule is correct. You may ignore the proof if you wish. The important thing is the rule itself.

- 1) (**Sum Rule**) The sum of two functions  $f$  and  $g$  with limit 0 at 0 has limit 0 at 0.

**Proof.** Given  $\varepsilon > 0$ , by making  $h \neq 0$  sufficiently small, you can make  $|f(h)| < \varepsilon/2$  and  $|g(h)| < \varepsilon/2$ , so by the triangle inequality,

$$|f(h) + g(h)| \leq |f(h)| + |g(h)| < \varepsilon/2 + \varepsilon/2 = \varepsilon. \quad \square$$

**EXAMPLE:**  $f(h) = h + 3h$  has limit 0 at 0.

- 2) (**Squeeze Rule**) If  $g$  is a function of  $h$  defined on some open interval about 0 except perhaps at 0, and  $f$  is a function with limit 0 at 0 such that  $|g(h)| \leq |f(h)|$  on some open interval about 0 except perhaps at 0, then  $g$  has limit 0 at 0.

**Proof.** By making  $h \neq 0$  sufficiently small, you can make  $|f(h)| < \varepsilon$ , and thus make  $|g(h)| < \varepsilon$ .  $\square$

**EXAMPLE:**  $E(h) = h^2$  has limit 0 at 0 since on  $(-1, 1)$ ,  $h^2 \leq |h|$ .

- 3) (**Product Rule**) Let  $g$  be a function of  $h$  such that for all values of  $h$  in some open interval about 0 except perhaps at 0, we have  $|g(h)| \leq c$  for some positive constant  $c$  (for example,  $g$  may itself have limit 0 at 0.) Let  $f$  have limit 0 at 0. Then  $f \cdot g$  has limit 0 at 0.

**Proof.** Since  $|f(h) \cdot g(h)| \leq c \cdot |f(h)|$  on some open interval about 0, by the squeeze rule, we need only show that  $c \cdot |f(h)|$  has limit 0 at 0. For this, we note that for any  $\varepsilon > 0$ , we may choose  $\delta$  so small that when  $h \neq 0$  is in the interval  $(-\delta, \delta)$ , we have  $|f(h)| < \varepsilon/c$ , whence  $c \cdot |f(h)| < c \cdot (\varepsilon/c) = \varepsilon$ .  $\square$

**EXAMPLE:**  $f(h) = h \cdot \cos(h)$  has limit 0 at 0.

4) (**Composition Rule**) If  $f$  and  $g$  have limit 0 at 0 and  $g(0) = 0$ , then  $g(f(h))$  has limit 0 at 0.

**Proof.** Given  $\varepsilon > 0$ , there is a  $\delta > 0$  such that if  $k$  is in the interval  $(-\delta, \delta)$ , then  $|g(k)| < \varepsilon$ . Also, there is also a  $\gamma > 0$  such that if  $h \neq 0$  is in the interval  $(-\gamma, \gamma)$ , then  $f(h)$  is in the interval  $(-\delta, \delta)$ , and so we have  $|g(f(h))| < \varepsilon$ .  $\square$

**EXAMPLE:**  $F(h) = (h \cdot \cos(h))^2$ .

5) Also we need to know that the constant function  $f(h) \equiv 0$  has limit 0 at 0.

#### 10. ONE-SIDED LIMITS AT 0

The above definition and resulting rules hold if we let  $h = \Delta x$  approach 0 only through positive values or only through negative values. That is, we only consider  $h > 0$  or we only consider  $h < 0$ . The notation for the first case is  $\lim_{h \rightarrow 0^+}$  and the notation for the second case is  $\lim_{h \rightarrow 0^-}$ .