

1. HOMEWORK DUE THURSDAY, AUGUST 31 AT 9 A.M.

Section 1.3: #4, 8, 10, 12, 14, and 16.

Section 1.4: #14, 16, 22, and 24.

This second homework with graphs is now online as HW2NewAug31. You will go over this homework this week, but **you can still work it until noon on Sunday**. Some people guessed and got good scores, but to get the benefit of the homework, I am asking that it be done with the graphs. If you somehow already did this homework with the graphs, let me know and I will give you the score that was recorded before the system went down. I hope for smooth sailing from now on.

2. HOMEWORK 3 DUE TUESDAY, SEPTEMBER 5 AT 9 A.M.

Section 2.1: #4, 8, 18, 22, 26.

Section 2.2: #4, 14, 22, 30, 32, 34, 36.

Give exact answers unless asked for an approximation.

Don't put in equal signs unless asked for them.

3. HOMEWORK 4 DUE THURSDAY, SEPTEMBER 7 AT 9 A.M.

Section 2.1: #10, 16.

Section 2.2: #24, 38. Give exact answer.

Section 2.3: #26, 28, 32, 36, 42, 44.

Don't put in equal signs unless asked for them.

4. WRITTEN PROBLEM FOR NEXT WEEK

Show that $\lim_{x \rightarrow 2} x^2 + 5x - 7 = 7$ **using the facts** that $\lim_{h \rightarrow 0} h^2 = 0$ and $\lim_{h \rightarrow 0} 9h = 0$.

You must use these facts in your demonstration to get credit for the problem.

Hint: You need to show that $\lim_{h \rightarrow 0} [(2+h)^2 + 5(2+h) - 14] = 0$.

5. RULES FOR FUNCTIONS WITH LIMIT 0 AT 0.

I repeat these rules, and give the proof now for the last two rules.

You may ignore the proof if you wish. The important thing is the rule itself.

- 1) (**Sum Rule**) The sum of two functions f and g with limit 0 at 0 has limit 0 at 0.
- 2) (**Squeeze Rule**) If g is a function of h defined on some open interval about 0 except perhaps at 0, and f is a function with limit 0 at 0 such that $|g(h)| \leq |f(h)|$ on some open interval about 0 except perhaps at 0, then g has limit 0 at 0.

- 3) (**Product Rule**) Let g be a function of h such that for all values of h in some open interval about 0 except perhaps at 0, we have $|g(h)| \leq c$ for some positive constant c (for example, g may itself have limit 0 at 0 or be a constant function.) Let f have limit 0 at 0. Then $f \cdot g$ has limit 0 at 0.
- 4) (**Composition Rule**) If f and g have limit 0 at 0 and $g(0) = 0$, then $g(f(h))$ has limit 0 at 0.

Proof. Given $\varepsilon > 0$, there is a $\delta > 0$ such that if k is in the interval $(-\delta, \delta)$, then $|g(k)| < \varepsilon$. Also, there is also a $\gamma > 0$ such that if $h \neq 0$ is in the interval $(-\gamma, \gamma)$, then $f(h)$ is in the interval $(-\delta, \delta)$, and so we have $|g(f(h))| < \varepsilon$. \square

EXAMPLE: $F(h) = (h \cdot \cos(h))^2$.

- 5) (**Constant Rule**) If C is a constant such as 0, and for all h in some interval about 0 except perhaps 0 we have $f(h) = C$, then $f(h) - C$ has limit 0 at 0.

Proof. $|f(h) - C|$ is always smaller than any $\varepsilon > 0$.

6. ONE-SIDED LIMITS AT 0

The above definition and resulting rules hold if we let $h = \Delta x$ approach 0 only through positive values or only through negative values. That is, we only consider $h > 0$ or we only consider $h < 0$. The notation for the first case is $\lim_{h \rightarrow 0+}$ and the notation for the second case is $\lim_{h \rightarrow 0-}$.

7. FINDING GENERAL LIMITS USING LIMITS 0 AT 0

Soon, all limits in this course will be $\lim_{h \rightarrow 0}$ or $\lim_{\Delta x \rightarrow 0}$. To work with $\lim_{x \rightarrow a}$, replace x with $a + h$ and then find $\lim_{h \rightarrow 0}$. For example,

$$\begin{aligned} \lim_{x \rightarrow 3} (x^2 - 2x + 9) &= \lim_{h \rightarrow 0} [(3 + h)^2 - 2(3 + h) + 9] \\ &= \lim_{h \rightarrow 0} [9 + 6h + h^2 - 6 - 2h + 9] = 12. \end{aligned}$$

As we will see, since the function is a polynomial, this is the same result as replacing the variable x with 3.

To show that $\lim_{x \rightarrow a} f(x) = L$, show that $\lim_{x \rightarrow a} [f(x) - L] = 0$, that is, $\lim_{h \rightarrow 0} [f(a + h) - L] = 0$. Suppose $\lim_{h \rightarrow 0} f(a + h) = L$. Let $E(h) = f(a + h) - L$. Then $\lim_{h \rightarrow 0} E(h) = 0$. It follows that $f(a + h) = L + E(h)$, where E is a function having limit 0 as the input h goes to 0. We can use this to get the rules for general limits. That is $\lim_{x \rightarrow a} f(x) = L$ if and only if for all small h except perhaps 0, $f(a + h) = L + E(h)$ where $\lim_{h \rightarrow 0} E(h) = 0$.

8. RULES FOR LIMITS

Theorem 1 [Constant Law]. *If for some constant C , $f(x) = C$ for all x , then at a point a , $\lim_{x \rightarrow a} f(x) = C$.*

Proof. $f(a + h) - C$ is the constant function 0, which has limit 0 at $h = 0$. \square

Theorem 2. *A limit is unique. That if, $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} f(x) = M$, then $L = M$.*

Proof. By assumption, for all nonzero h in some open interval about 0, $f(a + h) = L + E(h) = M + F(h)$ where E and F both have limit 0 at 0. It follows that $L - M = F(h) - E(h)$. Since the left hand side is constant and the right hand side has limit 0 at 0, $L - M$ is the constant 0, that is, $L = M$.

Theorem 3. *If $f(x) = x$ for all x , then $\lim_{x \rightarrow a} f(x) = a$.*

Proof. $f(a + h) - a = (a + h) - a = h$, which has limit 0 at 0. \square

Theorem 4 [Sum, Product, & Quotient Law]. *If $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$, then*

$$\lim_{x \rightarrow a} f(x) \pm g(x) = L \pm M,$$

$$\lim_{x \rightarrow a} f(x) \cdot g(x) = L \cdot M,$$

and if $M \neq 0$,

$$\lim_{x \rightarrow a} f(x)/g(x) = L/M.$$

For the quotient, if $M = 0$ and $L \neq 0$, the limit does not exist; if $M = 0$ and $L = 0$ more work is needed to see if the limit exists.

REMARK: For the above theorem, we say that the limit of a sum is the sum of the limits and the limit of a product is the product of the limits. We also say that the limit of a quotient is the quotients of the limits provided the denominator is not zero. To see what might happen with limit 0/0, consider the functions $y = x^2/x$, $y = x/x$, $y = x/x^2$.

Proof. We are assuming that for all $h \neq 0$ in some interval about 0,

$$f(a + h) = L + E(h) \quad \text{and} \quad g(a + h) = M + F(h)$$

where E and F have limit 0 at 0. We will work in an interval for which this is true. The first rule with $+$, called the addition rule, follows from the fact that

$$f(a + h) + g(a + h) = (L + M) + E(h) + F(h).$$

which has limit $L + M$ as h approaches 0. A similar proof holds for subtraction.

The second rule, called the product rule, follows from the fact that

$$\begin{aligned} f(a+h) \cdot g(a+h) &= (L + E(h)) \cdot (M + F(h)) \\ &= L \cdot M + L \cdot F(h) + M \cdot E(h) + E(h) \cdot F(h), \end{aligned}$$

the last three terms have limit 0 at 0. The last rule, called the quotient rule, follows from the product rule if we just show that for $M \neq 0$,

$$\lim_{x \rightarrow a} \frac{1}{g(x)} = \frac{1}{M}.$$

But when $h \neq 0$, $g(a+h) = M + F(h)$ where F has limit 0 at 0, so

$$\frac{1}{g(a+h)} - \frac{1}{M} = \frac{M - (M + F(h))}{M(M + F(h))} = \left(\frac{-1}{M(M + F(h))} \right) \cdot F(h)$$

For small values of h , $|F(h)| \leq M/2$, and so $|M + F(h)| \geq M/2$, and so we have $|1/(M(M + F(h)))| \leq 2/M^2$. Thus, we have a function that is bounded near 0 times the function $F(h)$, which has limit 0 at 0. \square

Theorem 5. *If f is a polynomial, i.e., $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$, then $\lim_{x \rightarrow c} f(x) = f(c)$.*

Proof. This follows from the rules for constants, the rule for the function $f(x) = x$, and the above rules for addition and multiplication.

EXAMPLE: $\lim_{x \rightarrow 2} 3x^3 - 8x^2 + 2x - 5 = 24 - 32 + 4 - 5 = -9$.

Corollary 6. *If f is a rational function, that is the ratio of two polynomials $P(x)/Q(x)$, then at any point c where $Q(x)$ is not 0, $\lim_{x \rightarrow c} f(x) = P(c)/Q(c)$.*

Theorem 7 [Substitution Law]. *Suppose that $\lim_{x \rightarrow a} f(x) = L$, and for $z = g(y)$, $\lim_{y \rightarrow L} g(y) = g(L)$. Then $\lim_{x \rightarrow a} g(f(x)) = g(\lim_{x \rightarrow a} f(x)) = g(L)$. That is, we can interchange applying the function g with taking the limit at a .*

Proof. We are told that for small $h \neq 0$, $f(a+h) = L + E(h)$ where $E(h)$ has limit 0 at 0. We are also told that for small values of k including $k = 0$, $g(L+k) = g(L) + F(k)$, where $F(k)$ has limit 0 at 0 and is 0 at 0. Now, for $h \neq 0$,

$$g(f(a+h)) = g(L + E(h)) = g(L) + F(E(h)).$$

Since $F(E(h))$ has limit 0 at $h = 0$, we are done by the substitution rule for functions with limit 0 at 0.

EXAMPLE: $\lim_{x \rightarrow 2} (x^2 + 1)^7 = (\lim_{x \rightarrow 2} (x^2 + 1))^7 = 5^7$.

Theorem 8 [Root Law]. *If n is a positive integer and $a > 0$, then*

$$\lim_{x \rightarrow a} x^{\frac{1}{n}} = a^{\frac{1}{n}}.$$

EXAMPLE: Using the Root Law and the Substitution Law, we get

$$\lim_{x \rightarrow 3} \sqrt{x^2 - 2x + 1} = 2.$$

Note: The Root Law can be extended to all values of a if n is an odd positive integer. We omit the proof of the Root Law.

For $a > 0$, using the product rule and quotient rules, we also have $\lim_{x \rightarrow a} x^m = a^m$ for m a positive or negative integer. For a negative integer we have to avoid $a = 0$. This then gives us a power rule for m/n .

EXAMPLE: Here is another example that shows you can't just substitute to get the limit.

$$\lim_{x \rightarrow 9} \frac{x - 9}{\sqrt{x} - 3} = \lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{\sqrt{x} - 3} \cdot \frac{\sqrt{x} + 3}{1} = \lim_{x \rightarrow 9} (\sqrt{x} + 3) = 6.$$

To simplify, we can also multiply by 1 expressed as an appropriate ratio.

$$\begin{aligned} \lim_{x \rightarrow 9} \frac{x - 9}{\sqrt{x} - 3} &= \lim_{x \rightarrow 9} \frac{x - 9}{\sqrt{x} - 3} \cdot \frac{\sqrt{x} + 3}{\sqrt{x} + 3} \\ &= \lim_{x \rightarrow 9} \frac{x - 9}{x - 9} \cdot \frac{\sqrt{x} + 3}{1} = \lim_{x \rightarrow 9} (\sqrt{x} + 3) = 6. \end{aligned}$$

Theorem 9 [Squeeze Law]. *Suppose that $f(x) \leq g(x) \leq k(x)$ for all x in some open interval about a (except perhaps at a itself). Also suppose that $\lim_{x \rightarrow a} f(x) = L = \lim_{x \rightarrow a} k(x)$. Then $\lim_{x \rightarrow a} g(x) = L$.*

Proof. We are told that for small nonzero values of h ,

$$f(a + h) - L \leq g(a + h) - L \leq k(a + h) - L.$$

This means that

$$|g(a + h) - L| \leq |f(a + h) - L| + |k(a + h) - L|.$$

Since the right hand side is a function with limit 0 at 0, $g(a + h) - L$ has limit 0 at 0 by the simple squeeze law for functions with limit 0 at 0.

EXAMPLE: We know that for x close to 0 but not equal to 0,

$$1 - x^2 \leq 1 + x^2 \sin 1/x \leq 1 + x^2.$$

Since the left and right hand side have limit 1 at $x = 0$, $\lim_{x \rightarrow 0} (1 + x^2 \sin 1/x) = 1$.