

1. A TUTORING ROOM IS OPEN

7–9 p.m, Monday, Tuesday, Wednesday, Thursday, Room 140 Lincoln Hall.

2. HOMEWORK 4 DUE THURSDAY, SEPTEMBER 7 AT 9 A.M.

Section 2.1: #10, 16.

Section 2.2: #24, 38. Give exact answer.

Section 2.3: #26, 28, 32, 36, 42, 44.

Don't put in equal signs unless asked for them.

3. HOMEWORK 5 DUE TUESDAY, SEPTEMBER 12 AT 9 A.M.

Section 2.3: #2, 4, 6, 8, 10, 24.

Section 2.4: #20, 28, 30, 40, 42, 46.

4. WRITTEN PROBLEM FOR NEXT WEEK

Use the definition of the derivative (a.k.a., slope predictor function) to find the derivative of $f(x) = x^3$ at every x . **Do not** just invoke the power rule; show all your work.

5. EXAM FRIDAY, SEPTEMBER 15

On material through homework due Thursday, Sept. 14.

Section 4 (Liu Qi), Section 5 (Liu Qi) Section 6 (Michael Barrus), Section 8 (Scott Weaver) will take the exam in Room 314 Altgeld Hall.

Section 2 (Isaac Goldbring), Section 7 (Isaac Goldbring), Section 9 (Timothy LeSaulnier) will take the exam in Room 100 MSEB (Materials Science Engineering Building, North-West corner of Green and Mathews.)

Review Thursday September 14, Rooms 245, 443, 445 Altgeld Hall, 7-9 p.m.

6. MORE RULES FOR LIMITS

Theorem 1 [Squeeze Law]. *Suppose that $f(x) \leq g(x) \leq k(x)$ for all x in some open interval about a (except perhaps at a itself). Also suppose that $\lim_{x \rightarrow a} f(x) = L = \lim_{x \rightarrow a} k(x)$. Then $\lim_{x \rightarrow a} g(x) = L$.*

Proof. We are told that for small nonzero values of h ,

$$f(a+h) - L \leq g(a+h) - L \leq k(a+h) - L.$$

This means that

$$|g(a+h) - L| \leq |f(a+h) - L| + |k(a+h) - L|.$$

Since the right hand side is a function with limit 0 at 0, $g(a+h) - L$ has limit 0 at 0 by the simple squeeze law for functions with limit 0 at 0.

EXAMPLE: We know that for x close to 0 but not equal to 0,

$$1 - x^2 \leq 1 + x^2 \sin 1/x \leq 1 + x^2.$$

Since the left and right hand side have limit 1 at $x = 0$, $\lim_{x \rightarrow 0} (1 + x^2 \sin 1/x) = 1$.

7. SLOPE PREDICTOR FORMULA

Returning to the derivative, that is, the slope predictor formula, we are given a function f and a point $(a, f(a))$ on the graph. We want, if possible to find the slope of the tangent line at the point. This means, we want

$$\lim_{\Delta x \rightarrow 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}.$$

Notice that here we have a limit as the input approaches 0. For derivatives, we usually call the input that goes to 0, Δx instead of h .

EXAMPLE: For $f(x) = \sqrt{x}$, at any $a > 0$, we want for Δx small enough so that $a + \Delta x \geq 0$,

$$\begin{aligned} \lim_{\Delta x \rightarrow 0} \frac{\sqrt{a + \Delta x} - \sqrt{a}}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{a + \Delta x} - \sqrt{a}}{\Delta x} \cdot \frac{\sqrt{a + \Delta x} + \sqrt{a}}{\sqrt{a + \Delta x} + \sqrt{a}} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(a + \Delta x) - a}{\Delta x (\sqrt{a + \Delta x} + \sqrt{a})} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x / \Delta x}{\sqrt{a + \Delta x} + \sqrt{a}} = \frac{1}{2\sqrt{a}}. \end{aligned}$$

8. ONE-SIDED LIMITS

Often we want a limit as $x \rightarrow a$ just for values of $x > a$ or for values of $x < a$. For example, it makes no sense to speak about the limit of \sqrt{x} at 0 as $x \rightarrow 0$ from below since the square root is not defined for negative values. We write $\lim_{x \rightarrow a^+} f(x) = L$ if we are only concerned with values of $x > a$. That is, we look at $f(a + h)$ for only positive values of h . We write $\lim_{x \rightarrow a^-} f(x) = L$ if we are only concerned with values of $x < a$. That is, we look at $f(a + h)$ for only negative values of h . In the first case, the limit is called the right-hand limit of $f(x)$ at a , or the limit of $f(x)$ as x approaches a from above. In the second case, the limit is called the left-hand limit of $f(x)$ at a , or the limit of $f(x)$ as x approaches a from below.

If both the left and right-hand limits of $f(x)$ at a exist and are equal, then that common value is the limit of $f(x)$ at a . If both the left and right-hand limits of $f(x)$ at a exist but they are not equal, then the limit of $f(x)$ at a does not exist.

EXAMPLE: $\lim_{x \rightarrow 0} (|x|/x)$ does not exist since $|x|/x \equiv 1$ for $x > 0$ and $|x|/x \equiv -1$ for $x < 0$. This also says that the derivative or slope predictor function of

$f(x) = |x|$ does not exist at $x = 0$ since for $\Delta x \neq 0$,

$$\frac{f(0 + \Delta x) - f(0)}{\Delta x} = \frac{|\Delta x|}{\Delta x}.$$

Notice that the graph of $y = |\Delta x|$ has a sharp point at $x = 0$.

If a function f is not defined for values of $x < a$, then we say the left hand limit does not exist. If a function f is not defined for values of $x > 0$, then we say the right hand limit does not exist.

EXAMPLE: $\lim_{x \rightarrow 0^+} \sqrt{x} = 0$, but $\lim_{x \rightarrow 0^-} \sqrt{x} = 0$ does not exist.

9. TRIG LIMITS

First, we see geometrically using the unit circle (i.e., $r = 1$) that

$$\lim_{\theta \rightarrow 0} \sin \theta = 0 \quad \text{and} \quad \lim_{\theta \rightarrow 0} \cos \theta = 1.$$

Recall that for a region cut off by an angle θ in the disc about the origin of radius 1, the area $A(\theta)$ is found by noting that $A(\theta)/A(2\pi) = A(\theta)/\pi = \theta/2\pi$, that is $A(\theta) = \theta/2$. It follows that if you consider the areas of the three regions shown on Page 82 of your text in Figure 2.3.21, we have for $0 < \theta < \pi/2$,

$$\frac{1}{2} \sin \theta \cos \theta < \frac{1}{2} \theta < \frac{1}{2} \tan \theta.$$

Multiplying by $2/\sin \theta$, we have

$$\cos \theta < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}.$$

Inverting and noting that $1/(1/\cos \theta) = \cos \theta$, we have

$$\cos \theta < \frac{\sin \theta}{\theta} < \frac{1}{\cos \theta}.$$

For $-\pi/2 < \theta < 0$, the same inequality holds since $\sin \theta/\theta = \sin(-\theta)/(-\theta)$ and $\cos(-\theta) = \cos \theta$. Since $\lim_{\theta \rightarrow 0} \cos \theta = 1$ and $\lim_{\theta \rightarrow 0} \frac{1}{\cos \theta} = 1$, it follows from the squeeze law that

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1.$$

10. TRIG IDENTITIES YOU WILL NEED

You will need the law of cosines ($c^2 = a^2 + b^2 - 2ab \cos C$ etc.), which is a generalization of the Pythagorean Theorem, and also the following:

$$\sin^2 \theta + \cos^2 \theta = 1, \quad \tan^2 \theta + 1 = \sec^2 \theta, \quad \sec^2 \theta - 1 = \tan^2 \theta$$

$$\sin(\theta + \phi) = \sin \theta \cos \phi + \cos \theta \sin \phi$$

$$\cos(\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi$$

$$\sin(-\theta) = -\sin \theta \quad \cos(-\theta) = \cos \theta$$

$$\cos \theta = \sin\left(\theta + \frac{\pi}{2}\right), \quad \sin \theta = \cos\left(\theta - \frac{\pi}{2}\right).$$

11. TRIG LIMITS

We have shown that

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1.$$

APPLICATION:

$$\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta} = 0.$$

To see this, we apply the limit laws for a product, quotient, and sum after noting that

$$\frac{1 - \cos \theta}{\theta} = \frac{1 - \cos \theta}{\theta} \cdot \frac{1 + \cos \theta}{1 + \cos \theta} = \frac{\sin^2 \theta}{\theta(1 + \cos \theta)} = \frac{\sin \theta}{\theta} \cdot \frac{1}{1 + \cos \theta} \cdot \sin \theta.$$

EXAMPLE:

$$\lim_{\theta \rightarrow 0} \frac{\sin^3 \theta}{\theta^2} = \left(\lim_{\theta \rightarrow 0} \sin \theta\right) \cdot \left(\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta}\right) \cdot \left(\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta}\right) = 0.$$

EXAMPLE:

$$\lim_{x \rightarrow 0} \frac{\sin 5x}{\sin 3x} = \frac{5}{3} \cdot \lim_{x \rightarrow 0} \left(\frac{3x}{\sin 3x} \cdot \frac{\sin 5x}{5x}\right) = \frac{5}{3} \cdot \lim_{x \rightarrow 0} \left(\frac{3x}{\sin 3x}\right) \cdot \lim_{x \rightarrow 0} \left(\frac{5x}{\sin 5x}\right) = \frac{5}{3}.$$

EXAMPLE:

$$\lim_{x \rightarrow 0} \frac{5x}{\tan 3x} = \frac{5}{3} \lim_{x \rightarrow 0} \frac{3x}{\tan 3x} = \frac{5}{3} \cdot \lim_{x \rightarrow 0} \frac{3x}{\sin 3x} \cdot \lim_{x \rightarrow 0} \cos(3x) = \frac{5}{3}.$$