

1. A TUTORING ROOM IS OPEN

7–9 p.m, Monday, Tuesday, Wednesday, Thursday, Room 140 Lincoln Hall.

2. HOMEWORK 5 DUE TUESDAY, SEPTEMBER 12 AT 9 A.M.

Section 2.3: #2, 4, 6, 8, 10, 24.

Section 2.4: #20, 28, 30, 40, 42, 46.

3. HOMEWORK 6 DUE THURSDAY, SEPTEMBER 14 AT 9 A.M.

Section 3.1: #4, 6, 14, 18, 22, 28.

Section 3.2: #2, 4, 12, 14, 18, 30, 38, 50.

4. WRITTEN PROBLEM FOR NEXT WEEK

Use the definition of the derivative (a.k.a., slope predictor function) to find the derivative of $f(x) = x^3$ at every x . **Do not** just invoke the power rule; show all your work.

5. EXAM FRIDAY, SEPTEMBER 15

Time: 11a.m.

On material through Section 3.2, the basic rules for differentiation.

Section 4 (Liu Qi), Section 5 (Liu Qi) Section 6 (Michael Barrus), Section 8 (Scott Weaver) will take the exam in Room 314 Altgeld Hall.

Section 2 (Isaac Goldbring), Section 7 (Isaac Goldbring), Section 9 (Timothy LeSaulnier) will take the exam in Room 100 MSEB (Materials Science Engineering Building, North-West corner of Green and Mathews.) People in these sections **must** go to this room and not Altgeld Hall to take the exam.

Everyone should by now know their discussion section and section instructor. You will need to enter that on your examination. Bring your U of I identity card to show when turning in the exam.

Review Thursday September 14, Rooms 245, 443, 445 Altgeld Hall, 7-9 p.m.

6. INFINITE “LIMITS” AND LIMITS AT INFINITY

We often say that $\lim_{x \rightarrow a} f(x) = +\infty$ at a point a if in an open interval about a , the values $f(x)$ are positive, and $\lim_{x \rightarrow a} 1/f(x) = 0$. A similar idea works for $\lim_{x \rightarrow a} f(x) = -\infty$. We also have right and left hand infinite limits.

EXAMPLE: The function $y = 1/x$ becomes unbounded at 0. For positive values of x , $1/x$ becomes larger and larger as $x \rightarrow 0+$. We say that the right hand limit at 0 is $+\infty$, although this is not a real number. For negative values of x , $1/x$ is negative, and the absolute value becomes larger and larger as $x \rightarrow 0-$. We say that the left hand limit at 0 is $-\infty$, although this is also not a real number. Some will say that

the limit of $1/|x|$ at 0 is $+\infty$ since the unbounded behavior is the same from above and below 0. Note that $\lim_{x \rightarrow 0} 1/(1/|x|) = 0$.

We will usually reserve the word limit to mean a real-valued limit.

7. CONTINUITY.

A function f defined in an open interval containing a point a is continuous at a if $\lim_{x \rightarrow a} f(x)$ exists and equals $f(a)$. That is, for sufficiently small values of h in an open interval about 0 including the value $h = 0$, $f(a + h) - f(a)$ has limit 0 at $h = 0$ and is 0 at $h = 0$. For $h = \Delta x$ and $\Delta y = f(a + \Delta x) - f(a)$, we can say that as Δx goes to 0, Δy goes to 0. We can also say that $f(a + \Delta x) = f(a) + E(\Delta x)$ where $E(\Delta x)$ has limit 0 at 0 and is 0 at 0. Intuitively, a function is continuous at a point if its graph doesn't break there.

EXAMPLE: $f(x) = \sqrt{|x|}$ is continuous at 0 since $f(0) = 0$, and the limit from either side is 0.

If f is only defined for values $x \geq a$, we replace the limit at a with the limit from above in the definition of continuity of f at a . Similarly, if f is only defined for values $x \leq b$, we replace the limit at a with the limit from below in the definition of continuity of f at b .

EXAMPLE: $f(x) = \sqrt{x}$ is continuous at 0 since $f(0) = 0$, f is not defined below 0, and the limit from above is 0.

Definition 1. We say a function is continuous if it is continuous at every point of its domain.

EXAMPLE: Show $y = f(x) = x^2$ is continuous at every point a in its domain. That is, show that the squaring function is a continuous function.

PROOF: Fix a . For all small values of h including 0, $(a + h)^2 = a^2 + (2ah + h^2)$ and $2ah + h^2$ has limit 0 at $h = 0$ and is 0 at $h = 0$.

EXAMPLE: $\sqrt{1 - x^2} = \sqrt{(1 + x)(1 - x)}$ is continuous on its domain $[-1, 1]$, but we only consider the limit from below at 1 and the limit from above at -1 .

If there is no limit at a point in the domain of a function, for example the function jumps, then the function is not continuous. If there is a limit at a point but the function has a different value than the limit at that point or the function is not defined at that point, then you can redefine the function to make it continuous.

EXAMPLE: Let $f(x) = 0$ for all $x \neq 0$ but set $f(0) = 1$. You can redefine f at 0 to take the value 0 so that f as it is redefined is continuous everywhere.

EXAMPLE: $f(x) = (x^2 - 4)/(x - 2)$ for $x \neq 2$. At $x = 2$, there is no value for the function, but the limit is 4. We can extend the definition of the function by setting $f(2) = 4$. It is then continuous everywhere.

8. COMBINATIONS OF CONTINUOUS FUNCTIONS

It is immediate that a constant function and the function given by $f(x) = x$ are continuous. That is, they are continuous at every point of their domain, which in this case is the whole real line. If f and g are arbitrary functions continuous at a , so are $f + g$, $f - g$, and $f \cdot g$. The last rule includes the case that f is constant. For example, $5 \cdot g$ is continuous at a . This all means that a polynomial is continuous on the real line. We also have f/g continuous at a unless $g(a) = 0$. If we have $g(a) = 0$ and $f(a) = 0$, we may be able to define the ratio at a to get continuity. We say f/g has a **removable discontinuity** at a .

EXAMPLE: $\frac{x^2-1}{x-1}$ is not defined at 1, but if we set the value at 1 equal to 2, we have continuity at 1 since $\lim_{x \rightarrow 1} \frac{x^2-1}{x-1} = 2$. That is, the function has a removable discontinuity at $x = 1$.

Theorem 2. *If f is continuous at a and g is continuous at $f(a)$, then $g(f(x))$ is continuous at a .*

Proof. By the substitution law we have

$$\lim_{x \rightarrow a} g(f(x)) = g(\lim_{x \rightarrow a} f(x)) = g(f(a)).$$

EXAMPLE: $\sqrt{x^2+1}$ is continuous at every real number.

9. CONTINUITY OF TRIG FUNCTIONS

We will show that the functions \sin and \cos are continuous at every point, that is, they are continuous functions on all of the real line. It follows by the quotient rule that the functions \tan and \sec are continuous at points where \cos is not 0 and \cot and \csc are continuous at points where \sin is not 0.

To see that \sin and \cos are each continuous at a point a , we note that for small values of h including $h = 0$,

$$\begin{aligned}\sin(a+h) &= \sin a \cos h + \cos a \sin h, \\ \cos(a+h) &= \cos a \cos h - \sin a \sin h.\end{aligned}$$

Since $\lim_{h \rightarrow 0} \cos h = 1$, and $\lim_{h \rightarrow 0} \sin h = 0$, $\lim_{h \rightarrow 0} \sin(a+h) = \sin a$ and $\lim_{h \rightarrow 0} \cos(a+h) = \cos a$.

10. INTERMEDIATE VALUE PROPERTY.

If f is continuous on an interval $[a, b]$ and for some number K , $f(a) < K < f(b)$, then for some c with $a < c < b$, $f(c) = K$.

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In words: If a continuous function on an interval takes any two values s and t , then it takes each value between s and t at some point in the interval.

PROOF. The second statement follows from the first by considering $-f$. That is, if f satisfies the hypotheses for the second statement, then $-f$ satisfies the hypotheses for the first statement with respect to $-K$, so for some c in (a, b) , $-f(c) = -K$, whence $f(c) = K$.

Here is the idea for the proof of the first statement: We find the point c , by cutting $[a, b]$ in half and choosing the bottom half if in that bottom half the function ever takes a value $\geq K$; otherwise we choose the top half. Repeating this process, we get a collapsing sequence of intervals with one point in all of them, and that point is c .

EXAMPLE. We know that $y = x^3$ has a root in $[-1, 1]$ because x^3 is a continuous function of x and it is negative at -1 , and positive at 1 .

EXAMPLE. For a continuous function f , We know that $f(x) = K$ someplace in the interval (a, b) if we know that $f(a) - K < 0$ and $f(b) - K > 0$.

EXAMPLE. The function $f(x) = x^5 - 3x^3 + 1$ is continuous, $f(0) = 1$, and $f(1) = -1$, so there is a point c in the open interval $(0, 1)$ where $f(c) = 0$.