

1. A TUTORING ROOM IS OPEN

7–9 p.m, Monday, Tuesday, Wednesday, Thursday, Room 140 Lincoln Hall.

2. HOMEWORK 5 DUE TUESDAY, SEPTEMBER 12 AT 9 A.M.

Section 2.3: #2, 4, 6, 8, 10, 24.

Section 2.4: #20, 28, 30, 40, 42, 46.

3. HOMEWORK 6 DUE THURSDAY, SEPTEMBER 14 AT 9 A.M.

Section 3.1: #4, 6, 14, 16, 18, 22, 28.

Section 3.2: #2, 4, 6, 14, 42.

4. WRITTEN PROBLEM FOR THIS WEEK

Use the definition of the derivative (a.k.a., slope predictor function) to find the derivative of $f(x) = x^3$ at every x . **Do not** just invoke the power rule; show all your work.

5. EXAM FRIDAY, SEPTEMBER 15

Time: 11a.m.

On material through velocity and acceleration and the basic rules for the differentiation of sums and products. This means in particular, the rules for differentiating polynomials and products of polynomials. These rules are in Section 3.2.

Section 4 (Liu Qi), Section 5 (Liu Qi) Section 6 (Michael Barrus), Section 8 (Scott Weaver) will take the exam in Room 314 Altgeld Hall.

Section 2 (Isaac Goldbring), Section 7 (Isaac Goldbring), Section 9 (Timothy LeSaulnier) will take the exam in Room 100 MSEB (Materials Science Engineering Building, North-West corner of Green and Mathews.) People in these sections **must** go to this room and not Altgeld Hall to take the exam.

Everyone should by now know their discussion section and section instructor. You will need to enter that on your examination. Bring your U of I identity card to show when turning in the exam.

Review Thursday September 14, Rooms 245, 443, 445 Altgeld Hall, 7-9 p.m.

6. RATE OF CHANGE OF ONE VARIABLE WITH RESPECT TO ANOTHER

If we are given a function $y = f(x)$ on some interval or the whole real line, we will try to obtain the “instantaneous rate of change” of y with respect to x at a point x_0 in the domain of f by looking at ratios

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

That is, we want to come up with some “limiting value” as Δx goes to 0. If we have such a value we will denote it by $f'(x_0)$. This gives us a new function f' , which we

call the derivative of f . The domain of f' consists of all point in the domain of f where the above limit exists. We will define the tangent line to the graph of f at the point $(x_0, f(x_0))$ as the line through the point $(x_0, f(x_0))$ with slope $f'(x_0)$.

We have seen that

$$\text{If } f(x) = ax^2 + bx + c \text{ then } f'(x) = 2ax + b.$$

Notice that while the graph of f is a parabola, the graph of the function that gives the rate of change of f at each point x is a straight line. In general, the derivative of a polynomial $P(x)$ has degree one lower than the degree of $P(x)$.

Let us look at what it means for a function f to have a derivative at x ? We need to see what happens when x is fixed and we add a small nonzero change to x . We consider the change in the output of f when divided by that small change in the input. Some choices for how we write that change in the input are h and Δx . Using h and setting $\Delta y = f(x+h) - f(x)$, suppose we have a limit; we then write $\lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} = f'(x)$. We have seen this means that

$$\lim_{h \rightarrow 0} \left(\frac{f(x+h) - f(x)}{h} - f'(x) \right) = 0.$$

Equivalently, there is a function $E(h)$ (which we know is $\frac{f(x+h)-f(x)}{h} - f'(x)$) such that

$$\frac{\Delta y}{h} = \frac{f(x+h) - f(x)}{h} = f'(x) + E(h)$$

and $\lim_{h \rightarrow 0} E(h) = 0$.

Now we multiply the equation for $\frac{\Delta y}{h}$ by h , and we get

$$\Delta y = f(x+h) - f(x) = f'(x) \cdot h + E(h) \cdot h.$$

Adding $y = f(x)$, we also have

$$y + \Delta y = f(x+h) = f(x) + f'(x) \cdot h + E(h) \cdot h.$$

This means that if we approximate the graph of f by a straight line through the point $(x, f(x))$ with slope $f'(x)$, the y value, $f(x) + f'(x) \cdot h$, on that tangent line at $x+h$ is a very good approximation to the actual value of f at $(x+h)$. For small h , the difference between that approximate value and the actual value is the very small quantity $E(h) \cdot h$. If h were “infinitesimal”, we could not see the difference between the tangent line and the curve within the interval $(x-h, x+h)$. Most of calculus is the exploitation of this fact. We will often use both the above formula for Δy and the formula for $y + \Delta y$.

EXAMPLE. If $y = f(x) = x^2$, then for small $\Delta x \neq 0$,

$$y + \Delta y = (x + \Delta x)^2 = x^2 + 2x \cdot \Delta x + \Delta x \cdot \Delta x$$

and

$$\Delta y = (y + \Delta y) - y = 2x \cdot \Delta x + \Delta x \cdot \Delta x,$$

The term $\Delta x \cdot \Delta x$ becomes much smaller than $2x \cdot \Delta x$ for values of Δx close to 0.

Note that in the general calculation of Δy , $\lim_{\Delta x \rightarrow 0} \Delta y = 0$. This means the following:

Theorem 1. *If a function has a derivative at a point, then it is continuous at that point.*

It is not true that continuity at a point implies the existence of a derivative. For example, $f(x) = |x|$ is continuous everywhere but does not have a derivative at $x = 0$. To see this, we note that $\lim_{h \rightarrow 0^-} \frac{|0+h|-|0|}{h} = \lim_{h \rightarrow 0^-} -1 = -1$, while $\lim_{h \rightarrow 0^+} \frac{|0+h|-|0|}{h} = \lim_{h \rightarrow 0^+} 1 = 1$. Thus no limit exists. One can see in this example that the function has a sharp point at 0, and so many lines just touch the graph at this point. It would not make sense to talk about a tangent line. This is typical of the behavior of a function continuous at a point that has no derivative at the point.

In general, we will be working with derivatives at points where the function is defined in an open interval about the point. We can define one sided derivatives, which we will use at the end points of domains of definitions, just as we defined one sided limits. That is, $f'_+(x) = \lim_{h \rightarrow 0^+} \frac{f(x+h)-f(x)}{h}$, and $f'_-(x) = \lim_{h \rightarrow 0^-} \frac{f(x+h)-f(x)}{h}$. If f is defined on an open interval containing x , the derivative exists at x if and only if each one sided derivative exists at x and the two one sided derivatives are equal.

7. NOTATION FOR DERIVATIVES

Let $y = f(x)$. We have written $f'(x)$ for the derivative at x and f' for the derivative function. Other notation which is often used is $\frac{dy}{dx}$ for the derivative function. For example, if $y = ax^2 + bx + c$, then $\frac{dy}{dx} = 2ax + b$. This is a convenient way of writing the derivative as a function of x . It is notation developed by Leibniz. He was thinking of the infinitesimal change dy in y , divided by the infinitesimal change dx in x . We will simply take $\frac{dy}{dx}$ as notation for the derivative. One may also think of it as something done to y , i.e., $\frac{d}{dx}(y)$. Some authors also write y' for the derivative. It is also very convenient at times to write $D_x y$ or if x is understood just Dy for the derivative. Maple uses the notation D_x to take a derivative with respect to x .

Example: Let $y = 3x^2 - 4x + 5$. Then $\frac{dy}{dx} = 6x - 4$. We can also write $D_x(3x^2 - 4x + 5) = 6x - 4$.

8. RATES OF CHANGE

We can use derivatives to measure the rate of change of one quantity with respect to another.

EXAMPLE: If A is the area of a square of side x , then $A = x^2$, and $\frac{dA}{dx} = 2x$. That is, the rate of change of the area is twice the length of the side.

EXAMPLE: The area A of a disc of radius r is πr^2 . It follows that $\frac{dA}{dr} = 2\pi r$. Notice this is the length of the circumference of the disk. When the radius of the disk is 7, if you make a small negative change $\frac{-1}{100}$ in the radius, the change in the area is approximately $\frac{-14\pi}{100}$. If the unit of measure is, for example, in feet, then this means that $\Delta A \approx \frac{-14\pi}{100} \text{ft}^2$.

Let us now consider the case when the independent variable is time, usually denoted by t .

VELOCITY: Let t denote time and $x(t)$ denote the algebraic position on the real line as a function of t . For example, we usually set $x(0) = 0$. If $x(2) = -1$, this means at time $t = 2$, you have traveled 1 unit to the left of your starting position. Now, **velocity** is $v(t) = dx/dt$. If the velocity is positive, you are traveling to the right. If the velocity is negative, you are traveling to the left.

SPEED: This is the absolute value of velocity. That is it is $|dx/dt|$.

ACCELERATION: This is the time rate of change of velocity; $a(t) = dv/dt$.

For a free falling body, we usually write y instead of x , and letting the up direction be positive, we write

$$a = \frac{dv}{dt} = -g,$$

where g is the acceleration due to gravity. That is, $g \approx 32 \text{ ft/sec/sec}$, or $g \approx 9.8 \text{ meters/sec/sec}$, or $g \approx 980 \text{ centimeters/sec/sec}$. We can also write $980 \text{ centimeters/sec}^2$.

We will see later that if $f' = g'$ on an interval, then for some constant c , $f = g + c$ on that interval. This means that for a free falling body and time starting at time 0, since $a(t) = -g$ for all times t for which the body is in free fall, we have

$$v(t) = -gt + v_0$$

Where $v_0 = v(0)$. It then follows that for all times t for which the body is in free fall,

$$y(t) = \frac{-1}{2}gt^2 + v_0t + y_0.$$

where $y_0 = y(0)$.

Notice that the object will reach its maximum height when the upward velocity stops being positive and changes to negative. That is, when $t = \frac{v_0}{g}$.

EXAMPLE. Assume a ball is thrown straight up from the ground with an initial velocity of 64 feet per second. Then $y_0 = 0$, and the ball will reach its maximum height in 2 seconds. That height is $\frac{-1}{2} \cdot 32 \cdot 4 + 64 \cdot 2 = 64$ feet.

9. RULES FOR FINDING DERIVATIVES

We have seen that the derivative of a constant function is 0. That is, if c is a constant, then $\frac{dc}{dx} = D_x c = 0$.

In the following, we will assume that f and g have a derivative at a point x . For $y = f(x)$, set $\Delta y = f(x + \Delta x) - f(x)$. For $z = g(x)$, set $\Delta z = g(x + \Delta x) - g(x)$. We will use the fact that $f(x + \Delta x) = y + \Delta y$ and $g(x + \Delta x) = z + \Delta z$. Now

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = f'(x), \quad \text{and} \quad \lim_{\Delta x \rightarrow 0} \frac{\Delta z}{\Delta x} = \frac{dz}{dx} = g'(x).$$

Theorem 2 [Sum Rule]. *If f and g have derivatives at x , then so does the sum $f + g$. Here,*

$$(f + g)'(x) = f'(x) + g'(x).$$

That is, if $y = f(x)$ and $z = g(x)$, then in the Leibniz notation

$$\frac{d(y + z)}{dx} = \frac{dy}{dx} + \frac{dz}{dx}.$$

EXAMPLE: $D_x[(3x^2 + 4x + 5) + (5x^2 - x + 6)] = (6x + 4) + (10x - 1)$.

Proof of Sum Rule.

$$\begin{aligned} & \frac{[f(x + \Delta x) + g(x + \Delta x)] - [f(x) + g(x)]}{\Delta x} \\ &= \frac{[y + \Delta y + z + \Delta z] - [y + z]}{\Delta x} = \frac{\Delta y}{\Delta x} + \frac{\Delta z}{\Delta x}. \end{aligned}$$

The rest follows from the fact that the limit of a sum is the sum of the limits.

Theorem 3 [Product Rule]. *If f and g have derivatives at x , then so does the product $f \cdot g$. Here,*

$$(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x).$$

That is, if $y = f(x)$ and $z = g(x)$, then

$$D(yz) = z \cdot Dy + y \cdot Dz, \quad \text{or} \quad \frac{d(yz)}{dx} = \frac{dy}{dx} \cdot z + y \cdot \frac{dz}{dx}.$$

EXAMPLE: Let $f(x) = 3x - 2$, and $g(x) = 2x + 4$. If we multiply, we get

$$(f \cdot g)(x) = 6x^2 + 8x - 8$$

and the derivative is $12x + 8$. On the other hand,

$$f'(x) \cdot g(x) + f(x) \cdot g'(x) = 3(2x + 4) + (3x - 2)(2) = 12x + 8.$$

EXAMPLE: If $f(x) = \alpha$, where α is a constant, then

$$D(\alpha \cdot z) = (D\alpha) \cdot z + \alpha \cdot Dz = 0 + \alpha \cdot Dz = \alpha \cdot Dz$$

For example, $D3x^2 = 3Dx^2 = 6x$.

Proof of Product Rule.

$$\begin{aligned} & \frac{f(x + \Delta x) \cdot g(x + \Delta x) - f(x) \cdot g(x)}{\Delta x} \\ = & \frac{(y + \Delta y)(z + \Delta z) - y \cdot z}{\Delta x} = \frac{z \cdot \Delta y + y \cdot \Delta z + \Delta y \cdot \Delta z}{\Delta x} \\ = & z \cdot \frac{\Delta y}{\Delta x} + y \cdot \frac{\Delta z}{\Delta x} + \Delta y \cdot \frac{\Delta z}{\Delta x}. \end{aligned}$$

The rest follows by using the rule that the limit of a sum is the sum of the limits, the limit of a product is the product of the limits, and the fact that y and z are constants at the fixed point x , and $\lim_{\Delta x \rightarrow 0} \Delta y = 0$.

To see what is going on here, we can consider a rectangle with sides y and z where both are functions of time t .

Theorem 4 [Power Rule]. *For any rational number r , $D_x x^r = r x^{r-1}$. In particular, $Dx = 1$ (that is, $Dx^1 = 1 \cdot x^0$).*

EXAMPLE: If $f(x) = x^7$, then $f'(x) = 7x^6$.

We will establish this rule for all natural numbers n , and come back to the more general case later. For $n = 1$, $Dx = 1$. Assume we know the rule is true for n . Then using the product rule, we have

$$Dx^{n+1} = D[x \cdot x^n] = 1 \cdot x^n + x \cdot (nx^{n-1}) = (n + 1)x^n.$$

That is, if the rule is true for n , then it is true for $n + 1$. Since the rule is true for 1, it must be true for 2. Since it is true for 2, it must be true for 3, etc. Thus it must be true for all positive natural numbers n . This is an informal example of what is called “mathematical induction”.

Theorem 5. *For a polynomial*

$$\begin{aligned} p(x) &= a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0, \\ p'(x) &= n a_n x^{n-1} + (n - 1) a_{n-1} x^{n-2} + \dots + 2 a_2 x + a_1. \end{aligned}$$

EXAMPLE: $D(3x^4 - 2x^3 + x^2 + 5x - 3) = 12x^3 - 6x^2 + 2x + 5$.