1. A Tutoring Room is Open

7–9 p.m, Monday, Tuesday, Wednesday, Thursday, Room 140 Lincoln Hall.

2. Homework 6 due Thursday, September 14 at 9 a.m.

Section 3.1: #4, 6, 14, 16, 18, 22, 28.

Section 3.2: #2, 4, 6, 14, 42.

3. Homework 7 due Tuesday, September 19 at 9 a.m.

Section 3.2: #8, 10, 12, 16 18, 20, 22 30, 36, 38, 44, 48.

4. Written Problem for next week

The position function of a particle moving in a horizontal straight line with motion to the right being positive is given by $x(t) = -2t^3 + 3t^2 + 12t - 7$ for all times $t \geq 0$. Find the acceleration, velocity, and speed of the particle at any nonnegative time t, and find the particle's position when its velocity is zero.

5. Exam Friday, September 15

Time: 11a.m.

On material through velocity and acceleration and the basic rules for the differentiation of sums and products. This means in particular, the rules for differentiating polynomials and products of polynomials. These rules are in Section 3.2.

Section 4 (Liu Qi), Section 5 (Liu Qi) Section 6 (Michael Barrus), Section 8 (Scott Weaver) will take the exam in Room 314 Altgeld Hall.

Section 2 (Isaac Goldbring), Section 7 (Isaac Goldbring), Section 9 (Timothy LeSaulnier) will take the exam in Room 100 MSEB (Materials Science Engineering Building, North-West corner of Green and Mathews.) People in these sections **must** go to this room and not Altgeld Hall to take the exam.

Everyone should by now know their discussion section and section instructor. You will need to enter that on your examination. Bring your U of I identity card to show when turning in the exam.

Review Thursday September 14, Rooms 245, 443, 445 Altgeld Hall, 7-9 p.m.

6. Rates of Change

Consider the case when the independent variable is time, usually denoted by t. Position, velocity, and acceleration on a line depend on which direction is taken as positive. For a falling body, up is usually taken as positive, and so the acceleration due to gravity is usually taken as negative, i.e., $-g$. If an object is then thrown up, the initial velocity is positive.

An object moving along a horizontal line with the right side of 0 being positive has negative acceleration if it is moving with positive velocity, i.e., to the right, and it is braking. If the object has negative velocity, i.e., it is moving to the left, and it is braking, the acceleration is positive. Speed is the absolute value of the velocity.

7. Rules for finding derivatives

We have seen that the derivative of a constant function is 0. That is, if c is a constant, then $\frac{dc}{dx} = D_x c = 0$.

In the following, we will assume that f and g have a derivative at a point x. For $y = f(x)$, set $\Delta y = f(x + \Delta x) - f(x)$. For $z = g(x)$, set $\Delta z = g(x + \Delta x) - g(x)$. We will use the fact that $f(x + \Delta x) = y + \Delta y$ and $g(x + \Delta x) = z + \Delta z$. Now

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = f'(x), \quad \text{and} \quad \lim_{\Delta x \to 0} \frac{\Delta z}{\Delta x} = \frac{dz}{dx} = g'(x).$$

Theorem 1 [Sum Rule]**.** *If f and g have derivatives at x, then so does the sum $f + g$. Here,*

$$(f + g)'(x) = f'(x) + g'(x).$$

That is, if $y = f(x)$ and $z = g(x)$, then in the Leibniz notation

$$\frac{d(y + z)}{dx} = \frac{dy}{dx} + \frac{dz}{dx}.$$

EXAMPLE: $D_x[(3x^2 + 4x + 5) + (5x^2 - x + 6)] = (6x + 4) + (10x - 1)$.

Proof of Sum Rule.

$$\frac{[f(x + \Delta x) + g(x + \Delta x)] - [f(x) + g(x)]}{\Delta x}$$

$$= \frac{[y + \Delta y + z + \Delta z] - [y + z]}{\Delta x} = \frac{\Delta y}{\Delta x} + \frac{\Delta z}{\Delta x}.$$

The rest follows from the fact that the limit of a sum is the sum of the limits.

Theorem 2 [Product Rule]**.** *If f and g have derivatives at x, then so does the product $f \cdot g$. Here,*

$$(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x).$$

That is, if $y = f(x)$ and $z = g(x)$, then

$$D\,(yz) = z \cdot Dy + y \cdot Dz, \quad \text{or} \quad \frac{d(yz)}{dx} = \frac{dy}{dx} \cdot z + y \cdot \frac{dz}{dx}.$$

EXAMPLE: Let $f(x) = 3x - 2$, and $g(x) = 2x + 4$. If we multiply, we get

$$(f \cdot g)(x) = 6x^2 + 8x - 8$$

and the derivative is $12x + 8$. On the other hand,

$$f'(x) \cdot g(x) + f(x) \cdot g'(x) = 3(2x + 4) + (3x - 2)(2) = 12x + 8.$$

EXAMPLE: If $f(x) = \alpha$, where α is a constant, then

$$D\,(\alpha \cdot z) = (D\alpha) \cdot z + \alpha \cdot Dz = 0 + \alpha \cdot Dz = \alpha \cdot Dz$$

For example, $D3x^2 = 3Dx^2 = 6x$.

Proof of Product Rule.

$$\begin{aligned} &\frac{f(x + \Delta x) \cdot g(x + \Delta x) - f(x) \cdot g(x)}{\Delta x} \\ = &\frac{(y + \Delta y)(z + \Delta z) - y \cdot z}{\Delta x} = \frac{z \cdot \Delta y + y \cdot \Delta z + \Delta y \cdot \Delta z}{\Delta x} \\ &= z \cdot \frac{\Delta y}{\Delta x} + y \cdot \frac{\Delta z}{\Delta x} + \Delta y \cdot \frac{\Delta z}{\Delta x}. \end{aligned}$$

The rest follows by using the rule that the limit of a sum is the sum of the limits, the limit of a product is the product of the limits, and the fact that y and z are constants at the fixed point x, and $\lim_{\Delta x \to 0} \Delta y = 0$.

To see what is going on here, we can consider a rectangle with sides y and z where both are functions of time t.

Theorem 3 [Power Rule]**.** *For any rational number r, $D_x x^r = rx^{r-1}$. In particular, $Dx = 1$ (that is, $Dx^1 = 1 \cdot x^0$).*

EXAMPLE: If $f(x) = x^7$, then $f'(x) = 7x^6$.

We will establish this rule for all natural numbers n, and come back to the more general case later. For $n = 1$, $Dx = 1$. Assume we know the rule is true for n. Then using the product rule, we have

$$Dx^{n+1} = D[x \cdot x^n] = 1 \cdot x^n + x \cdot (nx^{n-1}) = (n + 1)x^n.$$

That is, if the rule is true for n, then it is true for $n + 1$. Since the rule is true for 1, it must be true for 2. Since it is true for 2, it must be true for 3, etc. Thus it must be true for all positive natural numbers n. This is an informal example of what is called “mathematical induction”.

Theorem 4. *For a polynomial*

$$\begin{aligned} p(x) &= a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0, \\ p'(x) &= n a_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \ldots + 2a_2 x + a_1. \end{aligned}$$

EXAMPLE: $D(3x^4 - 2x^3 + x^2 + 5x - 3) = 12x^3 - 6x^2 + 2x + 5.$

Theorem 5 [The Reciprocal Rule]**.** *If g is differentiable at x and $g(x) \neq 0$, then*

$$D_x \frac{1}{g(x)} = D_x\, z^{-1} = \frac{-g'(x)}{[g(x)]^2} = \frac{-1}{z^2} \cdot \frac{dz}{dx}.$$

Proof.

$$\frac{\frac{1}{z+\Delta z} - \frac{1}{z}}{\Delta x} = \frac{\frac{-\Delta z}{z(z+\Delta z)}}{\Delta x} = \frac{-1}{z(z+\Delta z)} \cdot \frac{\Delta z}{\Delta x}.$$

Now we have assumed that $z = g(x) \neq 0$, and we know that $\lim_{\Delta x \to 0} \Delta z = 0$, so the limit of the above expression as $\Delta x \to 0$ is

$$\lim_{\Delta x \to 0} \frac{\frac{1}{z+\Delta z} - \frac{1}{z}}{\Delta x} = \frac{-1}{z^2} \cdot \frac{dz}{dx}.$$

EXAMPLE:

$$D_x \frac{1}{x^2 - 3x + 2} = \frac{-(2x-3)}{(x^2 - 3x + 2)^2}.$$

Theorem 6 [Power Rule for nonzero integers]**.** *For an integer $n \neq 0$,*

$$Dx^n = nx^{n-1}.$$

Proof. We have already seen this for positive integers. For a negative integer $n = -m$ where m is positive, we have

$$Dx^n = D\frac{1}{x^m} = \frac{-1}{x^{2m}} \cdot mx^{m-1} = -mx^{-m-1} = nx^{n-1}.$$

EXAMPLE: $Dx^{-7} = -7x^{-8}$.

Note that $Dx^0 = D1 = 0$.

Theorem 7 [Quotient Rule]**.** *If f and g are differentiable at x and $g(x) \neq 0$, then f/g is differentiable at x and*

$$D_x \frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

This is also written as follows

$$\frac{d}{dx}\left(\frac{y}{z}\right) = \frac{z \cdot \frac{dy}{dx} - y \cdot \frac{dz}{dx}}{z^2}$$

Proof. Since

$$\frac{f(x)}{g(x)} = f(x) \cdot \frac{1}{g(x)},$$

the Product Rule and Reciprocal Rule give the following

$$D\frac{f(x)}{g(x)} = f'(x) \cdot \frac{1}{g(x)} + f(x) \cdot \frac{-1}{[g(x)]^2} \cdot g'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

EXAMPLE: The derivative of $(3x^3 - 2x + 4)/(x^2 + 5x - 1)$ is

$$\frac{(9x^2 - 2)(x^2 + 5x - 1) - (3x^3 - 2x + 4)(2x + 5)}{(x^2 + 5x - 1)^2}.$$