1. A Tutoring Room is Open

7–9 p.m, Monday, Tuesday, Wednesday, Thursday, Room 140 Lincoln Hall.

2. Homework 7 due Tuesday, September 19 at 9 a.m.

Section 3.2: #8, 10, 12, 16 18, 20, 22 30, 36, 38, 44, 48.

3. Homework 8 due Thursday, September 21 at 9 a.m.

Section 3.3: #2, 8, 10, 14, 18, 30, 50, 52.

Section 3.4: #4, 12, 20, 26.

4. Written Problem for this week

The position function of a particle moving in a horizontal straight line with motion to the right being positive is given by $x(t) = -2t^3 + 3t^2 + 12t - 7$ for all times $t \geq 0$. Find the acceleration, velocity, and speed of the particle at any nonnegative time t, and find the particle's position when its velocity is zero.

5. Grade on Exam

≤ 34, F, 14. $[35, 39]$, D-, 13. $[40, 44]$, D, 29. $[45, 49]$, D+, 24. $[50, 54]$, C-, 21. $[55, 59]$, C, 21. $[60, 64]$, C+, 14. $[65, 69]$, B-, 17. $[70, 74]$, B, 21. $[75, 79]$, B+, 11. $[80, 84]$, A-, 7. $[85, 89]$, A, 10. $[90, 100]$, A+, 4.

6. Answers to Exam

1a) Find the largest possible domain for the function $f(x) = \sqrt{x^2 - 36}$.

Ans: $x \leq -6$ or $x \geq 6$. Do **NOT** write $6 \leq x \leq -6$; 6 is not smaller than -6.

1b) If $y = f(x) = x^3 - 6x + 5$, at what values of x is the tangent line to the graph horizontal?

Ans: $f'(x) = 3x^2 - 6 = 0$ when $x = \pm\sqrt{2}$.

1c) Write the equation of the line through the point $(-3, 2)$ parallel to the line $2y + 8x - 5 = 0$.

Ans: The line is given by $y = -4x + 5/2$; it has slope -4. The desired line is given by $y - 2 = -4(x + 3)$, or $y = -4x - 10$. To check, we see that this is a line with slope -4 and it goes through the point $(-3, 2)$.

2a) Find the center and radius of the circle $x^2 + y^2 - 4y - 5 = 0$. Ans: Completing the square, we have $x^2 + (y - 2)^2 - 4 - 5 = 0$. this is the equation of a circle with center $(0, 2)$ and radius 3.

2b) Find $\lim_{x \to 16} \frac{x-16}{\sqrt{x}-4}$. Hint: $x - 16 = (\sqrt{x} - 4)(\sqrt{x} + 4)$.

Ans: $\lim_{x\to 16} \frac{x-16}{\sqrt{x}-4} = \lim_{x\to 16} (\sqrt{x}+4) = 8$.

3a) Find $\lim_{x\to 0} \frac{\sin^2 4x}{7x^2}$.

Ans: $\lim_{x\to 0} \frac{\sin^2 4x}{7x^2} = \lim_{x\to 0} \left(\frac{16}{7}\cdot\frac{\sin 4x}{4x}\cdot\frac{\sin 4x}{4x}\right) = \frac{16}{7}$.

3b) Let $f(x) = \frac{3\sin x}{x}$ for $x < 0$, and let $f(x) = \frac{1-\cos x}{x} + A$ for $x > 0$. Find a value for A so that 0 is a removable discontinuity of f.

Ans: $\lim_{x\to 0-} \frac{3\sin x}{x} = 3$ and $\lim_{x\to 0+} \frac{1-\cos x}{x} = 0$, so we must set $A = 3$.

4a) There is a result that tells you that the equation $f(x) = x^5 - x^2 + 5x - 2 = 0$ has a solution in the interval between -1 and 1. Give the name of that result, or at least describe what it says. (Hint: What is $f(-1)$; what is $f(1)$?) Ans: $f(-1) = -9$, and $f(1) = 3$. Since f is continuous, it must take the value 0 someplace between -1 and 1 by the Intermediate Value Theorem.

4b) Let $f(x) = x^2 + 2x - 8$. We know that $\lim_{x\to 3} f(x) = 7$, and this means that $f(3+h) = 7 + E(h)$ where $\lim_{h\to 0} E(h) = \lim_{h\to 0} (f(3+h) - 7) = 0$. Write the equation expressing $E(h)$ as a function of h for this case. Ans: Note, your answer should be a polynomial in h.

$$\begin{aligned} f(3+h) &= (3+h)^2 + 2(3+h) - 8 \\ &= 9 + 6h + h^2 + 6 + 2h - 8 \\ &= 7 + \left(8h + h^2\right). \end{aligned}$$

so $E(h) = 8h + h^2$.

5a) **Use the product rule** to find the following derivative, show all your work and **do not simplify**: Here is the answer:

$$\begin{aligned} & D_x\ (3x^5 - 2x^2)(3x^4 + 7x + 2) \\ = \ & (15x^4 - 4x)(3x^4 + 7x + 2) + (3x^5 - 2x^2)\left(12x^3 + 7\right). \end{aligned}$$

5b) Find the equation of the tangent line to the graph of the function given by the equation $y = x^3 - 3x^2 + 2x + 2$ at the point $(1, 2)$.

Ans: $dy/dx = 3x^2 - 6x + 2$. at $x = 1$, this is $3 - 6 + 2 = -1$. The equation of the tangent line is $y - 2 = -(x - 1)$, or $y = -x + 3$. As a check, this is the equation of the line with slope -1 through the point $(1, 2)$.

6a) The limit $\lim_{\Delta x \to 0} \frac{\sqrt{2+\Delta x}-\sqrt{2}}{\Delta x}$ is the derivative of what function at what point?

Ans: The function is $f(x) = \sqrt{x}$, the point is $x = 2$.

6b) Let f be a function with a derivative $f'(a)$ at a. We have seen that this means that for small nonzero values for Δx,

$$\frac{\Delta y}{\Delta x} = \frac{f(a+\Delta x) - f(a)}{\Delta x} = f'(a) + E(\Delta x)$$

where $\lim_{\Delta x \to 0} E(\Delta x) = 0$. Multiplying by Δx we get

$$\Delta y = f'(a) \cdot \Delta x + E(\Delta x) \cdot \Delta x.$$

Why does the last equation tell us that f is continuous at a?

Ans: The limit of Δy is 0 as Δx goes to 0.

7) Let $y = f(x) = |x|$. Let us try to calculate a derivative at $x = 0$. For $\Delta x > 0$, $\Delta y = |0 + \Delta x| - 0 = \Delta x$. For $\Delta x < 0$, $\Delta y = |0 + \Delta x| - 0 = -\Delta x$.

a) What is $\lim_{\Delta x \to 0+} \frac{\Delta y}{\Delta x}$? Ans: 1.

b) What is $\lim_{\Delta x \to 0-} \frac{\Delta y}{\Delta x}$? Ans: -1.

c) Why does the derivative of $f(x) = |x|$ not exist at $x = 0$? Ans: because the left and right hand limit of $\frac{\Delta y}{\Delta x}$ are not the same.

7. Sample Question for the quotient rule

Question: Evaluate the following derivative, show all your work and **do not simplify**.

$$D_x \frac{5x^3 - 2x + 7}{x^2 - 7x + 2}.$$

Ans:

$$D_x \frac{5x^3 - 2x + 7}{x^2 - 7x + 2} = \frac{(15x^2 - 2x)(x^2 - 7x + 2) - (5x^3 - 2x + 7)(2x - 7)}{(x^2 - 7x + 2)^2}$$

8. Power Rule

Theorem 1 [Power Rule for nonzero integers]**.** *For an integer $n \neq 0$,*

$$Dx^n = nx^{n-1}.$$

Proof. We have already seen this for positive integers. For a negative integer $n = -m$ where m is positive, we have

$$D_x x^n = D_x \frac{1}{x^m} = \frac{-1}{x^{2m}} \cdot mx^{m-1} = -mx^{-m-1} = nx^{n-1}.$$

EXAMPLE: $Dx^{-7} = -7x^{-8}$.

Note that $Dx^0 = D1 = 0$.

9. The Chain Rule

The chain rule deals with composite functions such as $(x^3 - 4x + 2)^{89}$. You would not want to multiply this out before differentiating. With the chain rule, you don't have to. It says that the derivative is

$$89 \cdot (x^3 - 4x + 2)^{88} \cdot (3x^2 - 4).$$

The idea is, you have y a function of a variable u, that is $y = g(u)$, and at the same time, u is a function of a variable x, that is $u = f(x)$. At a point x_0 where f is differentiable, $f(x_0) = u_0$. Assume g is differentiable at u_0. The chain rule says that the composition function $y = g(f(x))$ giving y as a function of x has a derivative at x_0 given by $g'(u_0) \cdot f'(x_0)$. That is, we have the following

Theorem 2 [Chain Rule]**.** *Given the above set up,*

$$Dg(f(x_0)) = g'(f(x_0)) \cdot f'(x_0).$$

This is sometimes written as

$$\frac{dy}{dx}|_{x=x_0} = \frac{dy}{du}|_{u=u_0} \cdot \frac{du}{dx}|_{x=x_0},$$

or just

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Note that cancellation of the "du's", although suggestive, does not make sense. One can think of this as follows: Let x be the position of a man, walking along beside a moving train. Let u denote the position of the train, and y the position of a bird flying over the train. Assume all three start at 0, and the train moves three times as fast as the man, so that $du/dx = 3$. Also assume the bird moves five times as fast as the train, so that $dy/du = 5$. Then the bird moves 15 times as fast as the man. That is

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Proof of Chain Rule. We know that for $\Delta u \neq 0$,

$$\frac{\Delta y}{\Delta u} = g'(u_0) + E(\Delta u)$$

or

$$\Delta y = g'(u_0) \cdot \Delta u + E(\Delta u) \cdot \Delta u,$$

where E has limit 0 at $\Delta u = 0$. We set $E(0) = 0$, and the last formula is still correct. Now we also know that

$$\Delta u = f'(x_0) \cdot \Delta x + F(\Delta x) \cdot \Delta x$$

where F has limit 0 at $\Delta x = 0$. This means that

$$\Delta y = g'(u_0)[f'(x_0) \cdot \Delta x + F(\Delta x) \cdot \Delta x] + E(\Delta u) \cdot [f'(x_0) \cdot \Delta x + F(\Delta x) \cdot \Delta x].$$

Dividing by $\Delta x \neq 0$, we have

$$\frac{\Delta y}{\Delta x} = g'(u_0) \cdot f'(x_0) + g'(u_0) \cdot F(\Delta x) + E(\Delta u) \cdot [f'(x_0) + F(\Delta x)].$$

Since Δu is has limit 0 at $\Delta x = 0$, $\lim_{\Delta x \to 0} E(\Delta u) = 0$. Also, if $\Delta u = 0$ for some $\Delta x \neq 0$, then $f'(x_0) + F(\Delta x) = 0$. It follows that all the terms on the right except the first have limit 0 as $\Delta x \to 0$. Thus the Chain Rule is proved. $\square$

An application of the chain rule is the **Generalized Power Rule.**

$$D_x[f(x)]^n = n[f(x)]^{n-1} \cdot f'(x).$$

Sample Question using the Chain Rule: Evaluate $D_x \frac{1}{(x^2-5x+2)^3}$. Here is the answer:

$$\begin{aligned} D_x \frac{1}{(x^2-5x+2)^3} &= D_x(x^2-5x+2)^{-3} = -3(x^2-5x+2)^{-4} \cdot (2x-5) \\ &= -\frac{3}{(x^2-5x+2)^4}(2x-5). \end{aligned}$$

Note: We can think of this as problem follows: $u = x^2 - 5x + 2$, and $y = u^{-3}$. Therefore,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -3(u)^{-4} \cdot (2x-5) = -\frac{3}{(x^2-5x+2)^4}(2x-5).$$

We have replaced the u with its value in terms of x for the final answer.

Example: We can use the chain rule to show that if g has a derivative at x, then $D_x(1/g(x)) = -g'(x)/(g(x))^2$. Ans: This is just the chain rule.