## 1. A Tutoring Room is Open

7–9 p.m, Monday, Tuesday, Wednesday, Thursday, Room 140 Lincoln Hall.

## 2. Homework 8 due Thursday, September 21 at 9 a.m.

Section 3.3: #2, 8, 10, 14, 18, 30, 50, 52.
Section 3.4: #4, 12, 20, 26.

## 3. Homework 9 due Tuesday, September 26 at 9 a.m.

Section 3.4: #24.
Section 3.5: #14, 20, 26, 28, 34, 38.
Section 3.6: #4, 8, 12, 20, 22 (give exact answers).

## 4. Written Problem for next week

A large cube of ice is decreasing in volume at a rate of 6 $\text{cm}^3$ per minute. Find the rate of change of the edge (it should be negative) when the volume is 64 $\text{cm}^3$? Show all your work.

**Note:** Every written homework problem is a sample question for exams.

## 5. Chain Rule

The idea is, you have $y$ a function of a variable $u$, that is $y = g(u)$, and at the same time, $u$ is a function of a variable $x$, that is $u = f(x)$. At a point $x_0$ where $f$ is differentiable, $f(x_0) = u_0$. Assume $g$ is differentiable at $u_0$. The chain rule says that the composition function $y = g(f(x))$ giving $y$ as a function of $x$ has a derivative at $x_0$ given by $g'(u_0) \cdot f'(x_0)$. That is, we have the following

**Theorem 1** [Chain Rule]**.** *Given the above set up,*

$$Dg(f(x_0)) = g'(f(x_0)) \cdot f'(x_0).$$

This is sometimes written as

$$\frac{dy}{dx}|_{x=x_0} = \frac{dy}{du}|_{u=u_0} \cdot \frac{du}{dx}|_{x=x_0},$$

or just

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

**Proof of Chain Rule.** We know that for $\Delta u \neq 0$,

$$\frac{\Delta y}{\Delta u} = g'(u_0) + E(\Delta u)$$

or

$$\Delta y = g'(u_0) \cdot \Delta u + E(\Delta u) \cdot \Delta u,$$

where $E$ has limit 0 at $\Delta u = 0$. We set $E(0) = 0$. Now we also know that

$$\Delta u = f'(x_0) \cdot \Delta x + F(\Delta x) \cdot \Delta x$$

where $F$ has limit 0 at $\Delta x = 0$. This means that

$$\Delta y = g'(u_0)[f'(x_0) \cdot \Delta x + F(\Delta x) \cdot \Delta x] + E(\Delta u) \cdot [f'(x_0) \cdot \Delta x + F(\Delta x) \cdot \Delta x].$$

Dividing by $\Delta x \neq 0$, we have

$$\frac{\Delta y}{\Delta x} = g'(u_0) \cdot f'(x_0) + g'(u_0) \cdot F(\Delta x) + E(\Delta u) \cdot [f'(x_0) + F(\Delta x)].$$

Since $\Delta u$ is has limit 0 at $\Delta x = 0$, $\lim_{\Delta x \to 0} E(\Delta u) = 0$. Also, if $\Delta u = 0$ for some $\Delta x \neq 0$, then $f'(x_0) + F(\Delta x) = 0$. It follows that all the terms on the right except the first have limit 0 as $\Delta x \to 0$. Thus the Chain Rule is proved. □

An application of the chain rule is the **Generalized Power Rule.**

$$D_x[f(x)]^n = n[f(x)]^{n-1} \cdot f'(x).$$

**Sample Question using the Chain Rule: Evaluate** $D_x \frac{1}{(x^2-5x+2)^3}$. Here is the answer:

$$\begin{aligned} D_x \frac{1}{(x^2 - 5x + 2)^3} &= D_x(x^2 - 5x + 2)^{-3} = -3(x^2 - 5x + 2)^{-4} \cdot (2x - 5) \\ &= -\frac{3}{(x^2 - 5x + 2)^4} (2x - 5). \end{aligned}$$

**Note:** We can think of this as problem follows: $u = x^2 - 5x + 2$, and $y = u^{-3}$. Therefore,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -3(u)^{-4} \cdot (2x - 5) = -\frac{3}{(x^2 - 5x + 2)^4} (2x - 5).$$

We have replaced the $u$ with its value in terms of $x$ for the final answer.

**Example:** We can use the chain rule to show that if $g$ has a derivative at $x$, then $D_x(1/g(x)) = -g'(x)/(g(x))^2$. Ans: This is just the chain rule and the power rule for power $-1$.

## 6. Application of Chain Rule

You may be given values for some quantities and asked for the value of another when rates of change of composition functions are involved.

EXAMPLE: If the radius $r$ of a sphere is changing as a function of time $t$, then the volume of the sphere is a function of $r$ and also a function of $t$ by composition. That is, $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(r(t))^3$. This means that the rate of change of the volume as a function of $t$ is the product of its derivative as a function of $r$ times the derivative of $r$ with respect to time $t$. For example, if $r = t^2 + 3t$, then

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = 4\pi r^2 \cdot (2t+3) = 4\pi \left(t^2 + 3t\right)^2 \cdot (2t+3).$$

Another kind of problem might ask you for the rate of change of the radius when you are given the rate of change of the volume and the value of $r$.

**Sample Question:** Let $V$ be the volume of a sphere. Suppose that $dV/dt = 5\ \mathrm{cm}^3/\sec$ at all times $t$. What is $dr/dt$ when $r = 2$ cm? Here is the answer: Since $\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$

$$\frac{dr}{dt} = \frac{\frac{dV}{dt}}{\frac{dV}{dr}} = \frac{5}{4\pi r^2} = \frac{5}{16\pi}\ \mathrm{cm}/\sec.$$

**Note:** Do not apply the fact that at the instant in question $r = 2$ until you have differentiated. If $r = 2$ at all times, then $V$ is constant and has 0 derivative.

## 7. Differentiation of rational powers

We have seen that the derivative of $x^n = nx^{(n-1)}$ for any nonzero integer $n$. Recall that a rational number has the form $p/q$, where $p$ and $q$ are integers and $q \neq 0$.

**Theorem 2.** *For any rational number $r \neq 0$, if $y = x^r$, $\frac{dy}{dx} = rx^{(r-1)}$.*

**Proof.** Let $r = p/q$. Then $y = x^{p/q}$, or $y^q = x^p$. Recognizing $y$ as a function of $x$ and differentiating both sides with respect to $x$ we get $qy^{(q-1)}\frac{dy}{dx} = px^{(p-1)}$, or

$$\frac{dy}{dx} = \frac{p}{q}\frac{x^{(p-1)}}{y^{(q-1)}} = \frac{p}{q}\frac{x^{(p-1)}}{x^{p(q-1)/q}} = \frac{p}{q}x^{(p-1-p+p/q)} = rx^{r-1}.$$

**EXAMPLE:** For $y = \sqrt{x} = x^{1/2}$, $dy/dx = \frac{1}{2} \cdot \frac{1}{\sqrt{x}}$.

**Sample Question:** Let $y = x^{2/3}$. Use the fact that $y^3 = x^2$ to show that $dy/dx = (2/3)x^{-1/3}$. Here is the answer:

$$\begin{aligned} D_x y^3 &= 3y^2\frac{dy}{dx} = 2x \\ \frac{dy}{dx} &= \frac{2}{3}\frac{x}{y^2} = \frac{2}{3}\frac{x}{x^{4/3}} = \frac{2}{3}x^{-1/3}. \end{aligned}$$

## 8. Maxima and Minima

If $f(x_0) \geq f(x)$ for all values of $x$ in the domain of $f$, we say that $f$ takes an **absolute maximum** value at $x_0$. If $f(x_0) \leq f(x)$ for all values of $x$ in the domain of $f$, we say that $f$ takes an **absolute minimum** value at $x_0$.

We say that a function $f$ takes a **local maximum value** at a point $x_0$ in its domain if the value of $f$ at $x_0$ is greater or equal to the value at nearby points. There may be bigger values far away. To be exact, if $f$ is defined on $[x_0 - \delta, x_0 + \delta]$ for some $\delta > 0$, and for all $x$ just in this interval $f(x) \leq f(x_0)$, then we say that $f$ takes a local maximum value at $x_0$. There may be a larger value of $f$ outside of this interval. Similarly, if $f$ is only defined on $[a, b]$ or $[a, +\infty)$ and for some $\delta > 0$, $f(a)$ is a maximum value of $f$ for the interval $[a, a + \delta]$, then we say that $f$ takes a local maximum value at $a$. A similar definition works for $b$ if $f$ is only defined on $[a, b]$ or $(-\infty, b]$. Similar definitions define the idea of a **local minimum**. **Note:** Instead of "takes a maximum value", we may say "has a maximum value".

**Theorem 3.** *If $f$ is defined at least on an interval $(x-\delta, x+\delta)$ and takes a maximum or minimum value on that interval at $x$, then either $f'(x)$ does not exist, or if it exists it must be $0$.*

**Proof.** Consider $\Delta y = f(x + \Delta x) - f(x)$ for values of $\Delta x$ with $|\Delta x| < \delta$. If $f$ has a maximum value at $x$, then $\Delta y \leq 0$, so for $\Delta x > 0$, $\Delta y/\Delta x \leq 0$, and for $\Delta x < 0$, $\Delta y/\Delta x \geq 0$. The only number that can be a limit from both sides is 0. If $f$ has a minimum value at $x$, then $\Delta y \geq 0$, so for $\Delta x > 0$, $\Delta y/\Delta x \geq 0$, and for $\Delta x < 0$, $\Delta y/\Delta x \leq 0$. The only number that can be a limit from both sides is 0.□