1. A Tutoring Room is Open

7–9 p.m, Monday, Tuesday, Wednesday, Thursday, Room 140 Lincoln Hall.

2. Homework 9 due Tuesday, September 26 at 9 a.m.

Section 3.4: #24.

Section 3.5: #14, 20, 26, 28, 34, 38.

Section 3.6: #4, 8, 12, 20, 22 (give exact answers).

3. Homework 10 due Thursday, September 28 at 9 a.m.

Section 3.6: #2, 6, 38, 44, 46 (give exact answers).

Section 3.7: #8, 12, 16, 24, 36, 66, 74 (work in radians).

4. Written Problem for next week

A large cube of ice is decreasing in volume at a rate of 6 cm^3 per minute. Find the rate of change of the edge (it should be negative) when the volume is 64 cm^3? Show all your work.

Note: Every written homework problem is a sample question for exams.

5. More on Maxima and Minima

Theorem 1. *If f is defined at least on an interval $(x-\delta, x+\delta)$ and takes a maximum or minimum value on that interval at x, then either $f'(x)$ does not exist, or if it exists it must be* 0.

Definition 2. *Assume f is defined and continuous on an interval I. The* ***critical points*** *(or numbers) of f on I are listed as follows:*
1) *Endpoints of I if they belong to I.*
2) *Points of I where the derivative of f does not exist.*
3) *Points of I where the derivative exists and is* 0.

Note: You do not need to check if a point of I that is an endpoint of I satisfies 2 or 3; it is a critical point regardless.

Theorem 3. *Assume f is defined and continuous on an interval I. The local maximum and minimum values of f occur only at the critical points of f on the real line. An absolute maximum must be a local maximum. An absolute minimum must be a local minimum. That is, the set of critical points contains the set of points where the function takes local maximum and local minimum values, and that set contains the set of points where the function takes absolute maximum and absolute minimum values.*

Proof. Assume that a local maximum or minimum value of f occurs at a point $x_0 \in I$. Also assume that x_0 is not an endpoint of I and that $f'(x_0)$ exists. Since for some $\delta > 0$, f takes an absolute maximum or minimum value for the interval $[x_0 - \delta, x_0 + \delta]$ at the point x_0, we have $f'(x_0) = 0$.

Theorem 4. *A continuous function will* ***always*** *take both an absolute maximum value and an absolute minimum value on an interval of finite length containing its endpoints. (Such an interval is called a closed and bounded interval.)*

EXAMPLES:

a) On $[-1, 1]$, $f(x) = |x|$ takes its maximum value at -1 and 1, and its minimum value at 0, where the derivative does not exist.

b) On $[-1, 1]$, $g(x) = x^2$ takes its minimum value at 0 where the derivative is 0, and its maximum value at -1, and 1, which are endpoints.

c) On $[-1, 1]$, $h(x) = x^3$ takes its minimum value at -1 and its maximum value at 1. At 0, the derivative is 0, but 0 is not a local maximum or a local minimum value of h on the interval $[-1, 1]$.

METHOD: Given a continuous function f on a closed and bounded interval $[a, b]$, to find the absolute maximum and minimum values, which you know exist, you check the values of f, at each of the critical points. If the biggest value is $f(x_1)$ then f takes its maximum value at x_1. There may be more than one such point. If the smallest value is $f(x_2)$, then f takes its minimum value at x_2. There may be more than one such point.

Sample Problem: Let $f(x) = x^3 - 3x^2 + 3x - 2$ on the interval $[0, 2]$. List all critical points of f on this interval, and find the maximum and minimum values of $f(x)$ on this interval.

SOLUTION. The derivative f' exists at all points and has the value $3x^2 - 6x + 3$. To find where $f'(x) = 0$, we divide by 3 and solve the equation $x^2 - 2x + 1 = 0$. The only solution is $x = 1$. The critical points are $0, 1, 2$. Now $f(0) = -2$, $f(1) = -1$, $f(2) = 8 - 12 + 6 - 2 = 0$. Thus, for the interval $[0, 2]$, f takes an absolute minimum value -2 at 0 and f takes an absolute maximum value 0 at 2.

6. Shortcuts to determining absolute maxima and minima

Proposition 5. *Suppose f is continuous on $[a, b]$ and $f(a) = f(b) = 0$. If $f(x) \geq 0$ at all points x in $[a, b]$,* ***and*** *there is only one critical point in (a, b), then the absolute maximum value of f is taken at that critical point. If $f(x) \leq 0$ at all points x in $[a, b]$,* ***and*** *there is only one critical point in (a, b), then the absolute minimum value of f is taken at that critical point.*

Proposition 6. *If f is a nonnegative, continuous function on the whole real line and goes to 0 at $-\infty$ and if f also goes to 0 at $+\infty$, then there is an interval $[a, b]$ in which f takes a value bigger than any value outside $[a, b]$. The maximum of f, on $[a, b]$ is then the maximum of f, on the whole real line.*

EXAMPLE: For $f(x) = 1/(x^2 + 1)$, the maximum value is taken at $x = 0$.

Proposition 7. *If f is a nonnegative, continuous function on the whole real line and goes to $+\infty$ at $-\infty$ and it also goes to $+\infty$ at $+\infty$, then there is an interval $[a, b]$ in which f takes a value smaller than any value outside $[a, b]$. The minimum of f, on $[a, b]$ is then the minimum of f, on the whole real line.*

EXAMPLE: For $f(x) = x^2$, the minimum value is taken at $x = 0$..

7. Applied Maximum-Minimum Problems

In solving these problems, try to follow the five point path:

1) Find the quantity to be maximized and minimized, i.e., the dependent variable.

2) Find the appropriate independent variable, often by eliminating other variables. Pictures and careful labels are needed here. Express the dependent variable as a function (say f) of the independent variable. This includes finding the appropriate interval of definition. Usually you can include the endpoints, perhaps by allowing degenerate cases. These first two steps are the hard part.

3) Find all critical points of the function f. These are the endpoints of the interval of definition, points where the derivative does not exist, and points where the derivative exists and is 0.

4) From these critical points, you find the point or points where the function takes its maximum and minimum values and you evaluate those values.

5) Now you answer the question that was asked. You should check the plausibility of your answer.

If you are given an interval of definition which is infinite, you often can restrict the problem to a closed and bounded sub-interval to answer the question. Also, you can often simplify the problem by finding the extreme points of a related function.

Sample Problem. Find the point on the line $y = -x + 1$ that is closest to the origin $(0, 0)$.

Solution. First, the distance of any point (x, y) from the origin is $\sqrt{x^2 + y^2}$. We can simplify the problem if we minimize the square of the distance $x^2 + y^2$. Since the point is on the line, $y = -x + 1$, we want to minimize $f(x) = x^2 + (-x + 1)^2 = 2x^2 - 2x + 1$. We know that when $x = 0$, $f(x) = 1$, and when $x = 1$, $f(x) = 1$; moreover, geometric considerations show that any smaller value of f, must occur for $0 < x < 1$. Therefore, we want to find the minimum value of $f(x)$, on $[0, 1]$. This takes care of steps 1 and 2.

3) $f'(x)$ exists for all x, and $f'(x) = 4x - 2 = 0$, when $x = 1/2$. Thus the critical points are $x = 0$, $1/2$, and 1.

4) $f(0) = 1$, $f(1/2) = 1/2$, and $f(1) = 1$.

5) Now to answer the question, we must remember that we set f equal to the square of the distance. Therefore, the closest point to the origin on the line $y = -x+1$ is the point $(1/2, 1/2)$, and its distance from the origin is $1/\sqrt{2}$. There is no farthest point, the endpoints 0 and 1, were introduced so that we could reduce the problem to a closed and bounded interval.

Sample Problem: Find the maximum volume of a cylinder that can be inscribed in a sphere of radius R.

1 and 2) Let r be the radius of the cylinder, and h the height of the cylinder. The volume of the cylinder is what we want to maximize; it is given by the formula

$$V = \pi r^2 h.$$

Our problem is to use the fact that the cylinder is inscribed in the sphere to express V as a function just of r or just of h. Assuming the cylinder is symmetrically placed in the sphere, we draw a picture and find that

$$R^2 = r^2 + (\frac{h}{2})^2, \qquad \text{so} \qquad V = \pi h \left(R^2 - \frac{h^2}{4} \right).$$

The interval of definition for V as a function of h is $0 \leq h \leq 2R$; of course at the endpoints, the volume of the cylinder is 0.

3) The derivative $\frac{dV}{dh}$ exists at all points in the interval. Moreover,

$$\frac{dV}{dh} = \pi \left(R^2 - \frac{h^2}{4} \right) - \frac{1}{2}\pi h^2 = \pi \left(R^2 - \frac{3}{4}h^2 \right) = 0$$

when $h = \frac{2R}{\sqrt{3}}$. Therefore, the critical points are $h = 0$, $\frac{2R}{\sqrt{3}}$, and $2R$.

4) For $h = 0$, and $2R$, the volume is 0, so we must have our maximum value at $h = \frac{2R}{\sqrt{3}}$.

5) That maximum volume is

$$\begin{aligned} V &= \pi h \left(R^2 - \frac{h^2}{4} \right) = \pi \cdot \frac{2R}{\sqrt{3}} \cdot \left(R^2 - \frac{4R^2}{4 \cdot 3} \right) \\ &= \pi \cdot \frac{2R}{\sqrt{3}} \cdot \frac{2}{3} R^2 = \frac{4\sqrt{3}}{9} \cdot \pi R^3. \end{aligned}$$

8. DIFFERENTIATION OF TRIGONOMETRIC FUNCTIONS

Recall that

$$\lim_{h \to o} \frac{\sin h}{h} = 1, \quad \text{and} \quad \lim_{h \to 0} \frac{1 - \cos h}{h} = 0.$$

Theorem 8. *The functions* sin *and* cos *are differentiable on the whole real line, and*

$$D \sin x = \cos x, \qquad D \cos x = -\sin x.$$

Proof. For small values of $h \neq 0$,

$$\lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h \to 0} \frac{[\sin x \ \cos h \ + \cos x \ \sin h] - \sin x}{h}$$

$$= \lim_{h \to 0} \left(\cos x \ \cdot \frac{\sin h}{h} - \sin x \cdot \frac{1 - \cos h}{h} \right) = \cos x.$$

For the result for cos, we note that

$$D_x \cos x = D_x \sin(x + \frac{\pi}{2}) = \cos(x + \frac{\pi}{2}) = \cos(-x - \frac{\pi}{2})$$

$$= \sin(-x) = -\sin x. \quad \square$$

We now have the following rules for differentiation of trigonometric functions that follow from these two rules and the usual rules on products, quotients, etc. **You must know these.** Note that the minus signs go with the "co" functions.

$$D_x \sin x = \cos x, \ \ D_x \cos x = -\sin x$$

$$D_x \tan x = \sec^2 x, \qquad D_x \cot x = -\csc^2 x$$

$$D_x \sec x = \sec x \ \tan x, \qquad D_x \csc x = -\csc x \ \cot x.$$

These rules hold at all points x for which the functions and corresponding derivatives are defined.

Proofs:

$$D_x \tan x = D_x \frac{\sin x}{\cos x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

$$D_x \cot x = D_x \frac{\cos x}{\sin x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x.$$

$$D_x \sec x = D_x \frac{1}{\cos x} = \frac{-1}{\cos^2 x} \cdot (-\sin x) = \sec x \ \tan x.$$

$$D_x \csc x = D_x \frac{1}{\sin x} = \frac{-1}{\sin^2 x} \cdot (\cos x) = -\csc x \ \cot x.$$

Sample Problem: Find $D_x \sin \sqrt{x}$.

Solution:

$$D_x \sin \sqrt{x} = (\cos \sqrt{x}) \cdot \frac{1}{2\sqrt{x}}.$$