1. A Tutoring Room is Open

7–9 p.m, Monday, Tuesday, Wednesday, Thursday, Room 140 Lincoln Hall.

2. Homework 9 due Tuesday, September 26 at 9 a.m.

Section 3.4: #24.
Section 3.5: #14, 20, 26, 28, 34, 38.
Section 3.6: #4, 8, 12, 20, 22 (give exact answers).

3. Homework 10 due Thursday, September 28 at 9 a.m.

Section 3.6: #2, 6, 38, 44, 46 (give exact answers).
Section 3.7: #8, 12, 16, 24, 36, 66, 74 (work in radians).

4. Written Problem for this week

A large cube of ice is decreasing in volume at a rate of 6 cm^3 per minute. Find the rate of change of the edge (it should be negative) when the volume is 64 cm^3? Show all your work.

5. Differentiation of trigonometric functions

Recall that

$$\lim_{h\to o}\frac{\sin h}{h}=1, \quad\text{and}\quad \lim_{h\to 0}\frac{1-\cos h}{h}=0.$$

Theorem 1. *The functions* sin *and* cos *are differentiable on the whole real line, and*

$$D\sin x=\cos x, \qquad D\cos x=-\sin x.$$

Proof. For small values of $h\neq 0$,

$$\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}=\lim_{h\to 0}\frac{[\sin x\ \cos h\ +\cos x\ \sin h]-\sin x}{h}$$
$$=\lim_{h\to 0}\left(\cos x\ \cdot\frac{\sin h}{h}-\sin x\cdot\frac{1-\cos h}{h}\right)=\cos x.$$

For the result for cos, we note that

$$\begin{aligned} D_x\cos x &= D_x\sin(x+\frac{\pi}{2})=\cos(x+\frac{\pi}{2})=\cos(-x-\frac{\pi}{2})\\ &= \sin(-x)=-\sin x.\ \square\end{aligned}$$

We now have the following rules for differentiation of trigonometric functions that follow from these two rules and the usual rules on products, quotients, etc. **You must know these.** Note that the minus signs go with the "co" functions.

$$D_x\sin x=\cos x,\ \ D_x\cos x=-\sin x$$

$$D_x \tan x = \sec^2 x, \quad D_x \cot x = -\csc^2 x$$
$$D_x \sec x = \sec x \ \tan x, \quad D_x \csc x = -\csc x \ \cot x.$$

These rules hold at all points x for which the functions and corresponding derivatives are defined.

Proofs:

$$D_x \tan x = D_x \frac{\sin x}{\cos x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

$$D_x \cot x = D_x \frac{\cos x}{\sin x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x.$$

$$D_x \sec x = D_x \frac{1}{\cos x} = \frac{-1}{\cos^2 x} \cdot (-\sin x) = \sec x \ \tan x.$$

$$D_x \csc x = D_x \frac{1}{\sin x} = \frac{-1}{\sin^2 x} \cdot (\cos x) = -\csc x \ \cot x.$$

Sample Problem: Find $D_x \sin \sqrt{x}$.

Solution:

$$D_x \sin \sqrt{x} = (\cos \sqrt{x}) \cdot \frac{1}{2\sqrt{x}}.$$

6. INVERSE FUNCTIONS

For certain functions $y = f(x)$, it is also true that x is a function of y, i.e., $x = g(y)$. That is, $x = g(f(x))$ and $y = f(g(y))$, so each function undoes the other. You may have to limit the domains of f and g so that this works. We say that the functions f and g are **inverses** of each other. We can go from the graph of f to the graph of g, by rotating about the line $y = x$.

EXAMPLES:

$y = x^3$ and $x = y^{1/3}$ work as inverse functions for all real values of x and y.

$y = x^2$ works for all real values of x, but if we want an inverse, we must either take $x \geq 0$, or $x \leq 0$. For the first case, $x = \sqrt{y}$ is the inverse function,and for the second, $x = -\sqrt{y}$ is the inverse function.

A simple example is $y = f(x) = x + 1$ and $x = g(y) = y - 1$. Another example is $y = f(x) = 5x$, and $x = g(y) = y/5$. These work for all real values. An example that works for all nonzero values is $y = f(x) = 1/x$, and $x = g(y) = 1/y$. Another example where x must be restricted to all positive values and y can take any real value is given by $y = \log_{10} x$ and $x = 10^y$.

Let f be a differentiable function on an interval I. We will consider two special cases. The first is when f is strictly increasing on I; that is, for $x_1 < x_2$, $f(x_1) <$

$f(x_2)$. The second case is when f is strictly decreasing on I; that is, for $x_1 < x_2$, $f(x_1) > f(x_2)$. Since f is differentiable, it is continuous on I, so by the intermediate value theorem, the set

$$J = \{y : y = f(x),\, x \text{ in } I\}$$

is also an interval. Moreover, we can find an inverse function $g(y)$ on J.

Theorem 2. *The function g is differentiable at each point of J, and if $y_0 = f(x_0)$, then*

$$g'(y_0) = g'(f(x_0)) = \frac{1}{f'(x_0)}.$$

EXAMPLE: If $y = x^{\frac{1}{3}}$,

$$D_x x^{\frac{1}{3}} = \frac{1}{3}x^{-\frac{2}{3}} \text{ and } D_y y^3 = 3y^2 = 3(x^{\frac{1}{3}})^2 = 3x^{\frac{2}{3}} = \frac{1}{\frac{1}{3}x^{-\frac{2}{3}}}.$$

Proof. As $\Delta x \to 0$, $\Delta y \to 0$, and visa versa. Since

$$\frac{\Delta x}{\Delta y} = \frac{1}{\frac{\Delta y}{\Delta x}},$$

the result follows from the limit laws for quotients.

If what follows, we will often write $y = f(x)$ and $y = f^{-1}(x)$ for both the function and its inverse.

7. Inverse Trigonometric Functions

We will work with the inverses of trigonometric functions. The simplest is the inverse of tan, called arctan or $\tan^{-1}$. Notice the latter notation does not mean $1/\tan$. It means the function that undoes what the tangent function does. The Tangent function is an increasing function taking the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ to $(-\infty, +\infty)$. It then repeats itself on the interval $(\frac{\pi}{2}, \frac{3\pi}{2})$, $(-\frac{3\pi}{2}, -\frac{\pi}{2})$, etc. We choose to invert the function just on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. Therefore, the inverse is an increasing function taking $(-\infty, +\infty)$ to $(-\frac{\pi}{2}, \frac{\pi}{2})$. The $\arctan x$ is the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose tangent is x.

8. Derivative of arctan

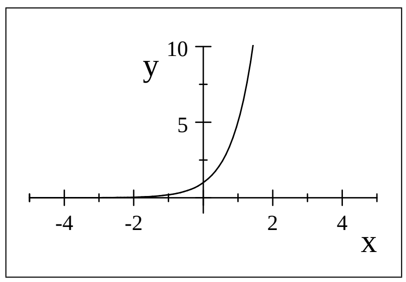
If $y = \arctan x$, then $x = \tan y$, $dx/dy = \sec^2 y = 1 + \tan^2 y = 1 + x^2$, so $dy/dx = 1/(1 + x^2)$. This is a very nice algebraic function that is the derivative of the trig function arctan.

9. Exponential Functions and Logarithmic Functions

We want to look at functions of the form a^x where a is a constant. We will assume that $a > 1$ so that as x increases, a^x increases. We know that $a^0 = 1$, that $a^1 = a$, and for an integer $n > 1$, $a^n = a \cdot a \cdots a$ where there are n factors. We also know that for a positive rational number, $\frac{n}{m}$, $a^{\frac{n}{m}}$ is the m^{th}-root of a^n, and $a^{-\frac{n}{m}} = 1/a^{\frac{n}{m}}$. This gives us a strictly increasing function of the rational numbers. We then can define a^x for all real numbers x by filling in the holes in the graph (that is, by taking "limits" over the rationals numbers.)

The graph of $y = a^x$ is that of an increasing, everywhere positive function of x. The limit as $x \to -\infty$ is 0, and the limit as $x \to +\infty$ is $+\infty$. The graph is concave up. That is, the graph is always below a straight line joining two points on the graph.

5^x



The following law of exponents holds for all rational numbers, and therefore by "taking limits", for all real numbers r and s.

$$a^{r+s} = a^r \cdot a^s, \quad (a^r)^s = a^{rs}, \quad a^{-r} = \frac{1}{a^r}, \quad (a \cdot b)^r = a^r \cdot b^r.$$

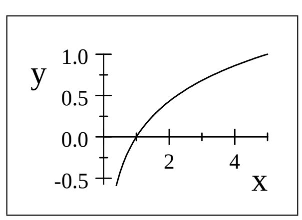
Here, we also take $b > 1$. (These rules, actually hold for all positive a and b.)

Common values for a are 2 and 10. For example, $2^3 = 8$, $10^3 = 1000$. We will later study the number $e = 2.718281828459....$ This is the base for "natural logs".

Corresponding to the function $y = a^x$ is the inverse function $y = \log_a x$. That is, y is the power it is necessary to raise a to get x. For example, $\log_2 8 = 3$, and $\log_{10} 1000 = 3$.

The graph of $y = \log_a x$ is only defined for $x > 0$. The graph is increasing, it is concave down, and it crosses the x-axis at $x = 1$ and takes the value 1 at a. The limit as $x \to 0$ is $-\infty$, and the limit as $x \to +\infty$ is $+\infty$.

$\log_5 x$



Here are rules for this function, which hold for all positive x and z, and any real number r.

$$\log_a (x \cdot z) = \log_a x + \log_a z, \quad \log_a x^r = r \cdot \log_a x.$$

The following rules are consequences of these two rules:

$$\log_a \frac{1}{x} = -\log_a x, \text{ and } \log_a \frac{x}{z} = \log_a x - \log_a z.$$

Sample Problem: Simplify $\log_{10} \frac{5000}{20}$. Ans:

$$\begin{aligned}\log_{10} \frac{5000}{20} &= \log_{10} 5000 - \log_{10} 20 \\ &= \log_{10} 5 + \log_{10} 1000 - \log_{10} 2 - \log_{10} 10 = 2 + \log{}_{10} 2.5.\end{aligned}$$

Alternatively, you can replace $\log_{10} \frac{5000}{20}$ with $\log_{10} \frac{500}{2}$ and go from there.

10. THE DERIVATIVE OF THE FUNCTION $y = a^x$

To calculate the derivative, if it exists, of the function $y = a^x$, $a > 1$, we look at the following difference quotient for $\Delta x \neq 0$:

$$\frac{a^{x+\Delta x} - a^x}{\Delta x} = \frac{a^x \cdot a^{\Delta x} - a^x}{\Delta x} = a^x \cdot \frac{a^{\Delta x} - 1}{\Delta x}.$$

Recall that $1 = a^0$. For the moment, we are going to assume that the derivative exists at $x = 0$, and call the derivative at 0 $m(a)$. That is,

$$m(a) = \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}.$$

It then follows that the derivative of $y = a^x$ exists at every point x and equals $a^x \cdot m(a)$. There is a real number $e \approx 2.71828\,18284\,6$ for which $m(a) = 1$. We will see later that we can also get e as the following limit using natural numbers n:

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n.$$

Try this calculation with your calculator using increasing values of natural numbers n.

Using the base e, we have the fact that

$$D_x\, e^x = e^x.$$

This fact makes the exponential function e^x one of the most useful functions in Science. You will from now on see many example of this function and its inverse function $\ln x$.

If u is a differentiable function of x, we then have by the chain rule,

$$D_x e^u = e^u \cdot \frac{du}{dx}.$$

Example:

$$D_x e^{e^{e^x}} = e^{e^{e^x}} \cdot e^{e^x} \cdot e^x.$$

Sample Question: Find the derivative of $e^{\sin x}$. Ans:

$$D_x e^{\sin x} = e^{\sin x} \cdot \cos x.$$

11. THE FUNCTION $\ln x$

The notation for $\log_e\ x$ is $\ln x$. This is defined only for $x > 0$. We know that $y = \ln x$ if and only if $x = e^y$. This means that

$$D_x \ln x = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{e^y} = \frac{1}{x}.$$

That is, the function given by $y = \ln x$ has as its derivative the function given by $y = \frac{1}{x}$.

If u is a differentiable function of x, then by the chain rule,

$$D_x \ln u = \frac{1}{u} \cdot \frac{du}{dx}.$$

EXAMPLE:

$$D_x \ln\left(\sqrt{x} + 5\right) = \frac{1}{\sqrt{x}+5} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2x + 10\sqrt{x}}, \qquad D_x \ln(\sin x)) = \cot x.$$

The function $\ln x$ is only defined for positive values of x. For example, the domain of $\ln(x-2)$ is all $x > 2$. We do have the following formula using the absolute value.

$$D_x \ln|x| = D_x x = \frac{1}{x} \text{ for } x > 0, \text{ and } D_x \ln|x| = D_x \ln(-x) = \frac{-1}{-x} = \frac{1}{x} \text{ for } x < 0.$$

That is, $D_x \ln|x| = \frac{1}{x}$ for $x \neq 0$.

Example:

$$D_x \ln\left(|\cos x|\right) = \frac{-\sin x}{\cos x} = -\tan x$$