1. A TUTORING ROOM IS OPEN

7–9 p.m, Monday, Tuesday, Wednesday, Thursday, Room 140 Lincoln Hall.

2. HOMEWORK 10 DUE THURSDAY, SEPTEMBER 28 AT 9 A.M.

Section 3.6: #2, 6, 38, 44, 46 (give exact answers).

Section 3.7: #8, 12, 16, 24, 36, 66, 74 (work in radians).

3. HOMEWORK 11 DUE TUESDAY, OCTOBER 3 AT 9 A.M.

Section 3.8: #12, 14, 16, 22, 30, 36, 40, 42, 50, 52, 56, 60. Put in Parentheses; for example $\cos(\ln(5x))$ not $\cos(\ln 5x)$.

4. WRITTEN PROBLEM FOR NEXT WEEK

You have a canal that makes a right angle turn. The width of the canal for the incoming leg is a, and the width of the canal for the outgoing leg is b. What is the longest narrow barge that can be moved around the turn?

Hint: Let θ be the angle formed by a line segment touching the inside corner and terminating at the outside walls of the canal; here the angel is formed by the line segment and the outside wall of the leg of width b. The length of the line segment is

$$L(\theta) = a\sec\theta + b\csc\theta.$$

As θ approaches 0 or $\frac{\pi}{2}$, the length of this line segment approaches $+\infty$. The maximum length of a barge that can make the turn is the minimum value of $L(\theta)$.

5. THE DERIVATIVE OF THE FUNCTION $y = a^x$

To calculate the derivative, if it exists, of the function $y = a^x$, $a > 1$, we look at the following difference quotient for $\Delta x \neq 0$:

$$\frac{a^{x+\Delta x} - a^x}{\Delta x} = \frac{a^x \cdot a^{\Delta x} - a^x}{\Delta x} = a^x \cdot \frac{a^{\Delta x} - 1}{\Delta x}.$$

Recall that $1 = a^0$. For the moment, we are going to assume that the derivative exists at $x = 0$, and call the derivative at 0 $m(a)$. That is,

$$m(a) = \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}.$$

It then follows that the derivative of $y = a^x$ exists at every point x and equals $a^x \cdot m(a)$. There is a real number $e \approx 2.71828\,18284\,6$ for which $m(a) = 1$. We will see later that we can also get e as the following limit using natural numbers n:

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

Try this calculation with your calculator using increasing values of natural numbers n.

Using the base e, we have the fact that

$$D_x\, e^x = e^x.$$

This fact makes the exponential function e^x one of the most useful functions in Science. You will from now on see many example of this function and its inverse function $\ln x$.

If u is a differentiable function of x, we then have by the chain rule,

$$D_x e^u = e^u \cdot \frac{du}{dx}.$$

Example:

$$D_x e^{e^{e^x}} = e^{e^{e^x}} \cdot e^{e^x} \cdot e^x.$$

Sample Question: Find the derivative of $e^{\sin x}$. Ans:

$$D_x e^{\sin x} = e^{\sin x} \cdot \cos x.$$

Sample Question: For what special value of a is $\lim_{\Delta x \to 0} \frac{a^{\Delta x}-1}{\Delta x} = 1$? **Ans:** $a = e$.

6. THE FUNCTION $\ln x$

The notation for $\log_e x$ is $\ln x$. This is defined only for $x > 0$. We know that $y = \ln x$ if and only if $x = e^y$. This means that

$$D_x \ln x = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{e^y} = \frac{1}{x}.$$

That is, the function given by $y = \ln x$ has as its derivative the function given by $y = \frac{1}{x}$.

If u is a differentiable function of x, then by the chain rule,

$$D_x \ln u = \frac{1}{u} \cdot \frac{du}{dx}.$$

EXAMPLE:

$$D_x \ln\left(\sqrt{x} + 5\right) = \frac{1}{\sqrt{x}+5} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2x + 10\sqrt{x}}, \qquad D_x \ln(\sin x)) = \cot x.$$

The function $\ln x$ is only defined for positive values of x. For example, the domain of $\ln(x-2)$ is all $x > 2$. We do have the following formula using the absolute value.

$$D_x \ln|x| = D_x x = \frac{1}{x} \text{ for } x > 0, \text{ and } D_x \ln|x| = D_x \ln(-x) = \frac{-1}{-x} = \frac{1}{x} \text{ for } x < 0.$$

That is, $D_x \ln|x| = \frac{1}{x}$ for $x \neq 0$.

Example:

$$D_x \ln(|\cos x|) = \frac{-\sin x}{\cos x} = -\tan x$$

7. RIGHT TRIANGLES

We discuss two right triangles, one with $\pi/4$, $\pi/4$ angles and the other with $\pi/6$, $\pi/3$ angles.

8. LOGARITHMIC DIFFERENTIATION

If a function has the form of one function raised to a power given by a second function, it often helps to take ln and then differentiate. That is, if

$$\begin{aligned} y &= f(x)^{g(x)}, \text{ then } \ln y = g(x)\ln f(x), \quad \text{so} \\ \frac{1}{y}\frac{dy}{dx} &= g'(x)\ln f(x) + g(x)\frac{f'(x)}{f(x)}. \end{aligned}$$

For example, if $y = a^x$, then we can differentiate in two ways. $y = e^{x\ln a}$, so $dy/dx = \ln a \cdot e^{x\ln a} = \ln a \cdot a^x$. Alternatively,

$$\ln y = x\ln a, \quad \text{so} \quad \frac{1}{y}\frac{dy}{dx} = \ln a, \quad \text{whence} \quad \frac{dy}{dx} = \ln a \cdot y = \ln a \cdot a^x.$$

If $y = x^{\sin x}$, then $\ln y = \sin x \cdot \ln x$, so

$$\frac{dy}{dx} = y\left(\cos x \cdot \ln x + \frac{\sin x}{x}\right) = x^{\sin x}\left(\cos x \cdot \ln x + \frac{\sin x}{x}\right).$$

9. A LIMIT THEOREM FOR e

This will not be done in class and will not be on an examination.

We want to quickly show that $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$. This will follow if we show that $\lim_{h\to 0}(1+h)^{\frac{1}{h}} = e$, and this will follow from the continuity of the strictly increasing continuous function ln if we show that $\lim_{h\to 0}\ln(1+h)^{\frac{1}{h}} = 1$. Now,

$$\lim_{h\to 0}\ln(1+h)^{\frac{1}{h}} = \lim_{h\to 0}\frac{\ln(1+h)}{h} = \lim_{h\to 0}\frac{\ln(1+h) - \ln 1}{h}.$$

The last limit is by definition the value of the derivative of $\ln x$ at $x = 1$, and this value is 1. That is,

$$\lim_{h\to 0}\ln(1+h)^{\frac{1}{h}} = 1,$$

and so we are done. Now for any $x \neq 0$,

$$\lim_{n\to\infty}\ln(1+\frac{x}{n})^n = \lim_{n\to\infty}\left(x\cdot\frac{n}{x}\cdot\ln(1+\frac{x}{n})\right) = x\cdot\lim_{h\to 0}(1+h)^{\frac{1}{h}} = x.$$

Composing with the continuous function exp, we see that for all values of x,

$$\lim_{n\to\infty}(1+\frac{x}{n})^n = e^x.$$

10. IMPLICIT DIFFERENTIATION

If f and g are two functions of x such that $f(x) = g(x)$ for all x in some open interval (a, b), then of course $f'(x) = g'(x)$ for all x in (a, b).

Often we are given an equation involving x and y and that equation determines y implicitly as a function of x on some open interval (a, b). For example,

$$x^2 + y^2 = 1 \tag{1}$$

determines y implicitly as a function of x on the interval $(-1, 1)$. We can either set $y = \sqrt{1-x^2}$ or $y = -\sqrt{1-x^2}$. If we know that $x = \frac{1}{\sqrt{2}}$, $y = \frac{1}{\sqrt{2}}$ is on the graph of the function we want, then we know that the first function is what we want. We then know that the derivative

$$\frac{dy}{dx} = \frac{-x}{\sqrt{1-x^2}},$$

and at $x = \frac{1}{\sqrt{2}}$, $y = \frac{1}{\sqrt{2}}$, $\frac{dy}{dx} = -1$. On the other hand, using the fact that y is a function of x, we can use the chain rule on Equation 1 to get

$$2x + 2y\frac{dy}{dx} = 0, \quad \text{whence} \quad \frac{dy}{dx} = \frac{-x}{y}$$

and so at the point $x = \frac{1}{\sqrt{2}}$, $y = \frac{1}{\sqrt{2}}$, we have $\frac{dy}{dx} = -1$.

The latter calculation is the idea of implicit differentiation. You are given an equation involving two variables and you assume that, at least on some interval, the equation makes one of the variables a function of the other. (In a later course, you will be given conditions to assure that this happens.) Often you are given a point on the graph of the implicitly defined function. You use the chain rule and sometimes the original equation to find the derivative of that function in terms of the two variables. If you know a point on the graph, you can then find the slope of the tangent line at that point.

Example: Find the equation of the tangent line to the curve

$$(x^2 + y^2)^3 = 8x^2y^2 \quad \text{at the point } (1, -1).$$

Note that the point is on the curve, as it should be. Assuming that in some open interval about $x = 1$ this equation defines y as a function of x, we have

$$3(x^2 + y^2)^2(2x + 2y \cdot \frac{dy}{dx}) = 16xy^2 + 16x^2y \cdot \frac{dy}{dx}.$$

Notice, we have not solved for y as a function of x. Now, making the substitution $x = 1$, $y = -1$, we have

$$3 \cdot 4 \cdot 2 \cdot (1 - \frac{dy}{dx}) = 16 - 16 \cdot \frac{dy}{dx}$$

or

$$3 \cdot (1 - \frac{dy}{dx}) = 2 \cdot (1 - \frac{dy}{dx}),$$

that is

$$\frac{dy}{dx} = 1.$$

Now, to find the equation of the tangent line we reserve the variables x, and y for the equation of that line. The slope is 1 and the line goes through the point $(1, -1)$. Therefore, the equation of the line is $y = x - 2$.

Sample Question: Write the equation of the tangent line to the graph of the curve $\frac{1}{1+x} + \frac{1}{1+y} = x$ at the point $(1, 1)$.

SOLUTION: Rather than solve for y as a function of x, we use implicit differentiation. That is, we think of y as a function of x and differentiate both sides of the equation with respect to x. This gives us

$$\frac{-1}{(1+x)^2} + \frac{-1}{(1+y)^2} \cdot \frac{dy}{dx} = 1.$$

Now we substitute the values for x and y and get $\frac{-1}{4} + \frac{-1}{4} \cdot \frac{dy}{dx} = 1$. That is, $\frac{dy}{dx} = \frac{5/4}{-1/4} = -5$. We are looking for the equation of the line with slope -5 going through the point $(1, 1)$. Now using x and y for the variables of the line, we get $y - 1 = -5(x - 1)$, or $y = -5x + 6$.

11. RELATED RATES OF CHANGE

For these problems, you are given information relating quantities which vary with time. That is, all the variables are dependent variables and time is the independent variable. You set up an equation expressing the given relationship. Rather than solving for one variable as a function of another, you use the chain rule to differentiate both sides of the equation as implicit functions time and you set the two derivatives equal to each other. That is, if $f(t) = g(t)$ for **all** times t relevant to the problem, then $f'(t) = g'(t)$. You then use the given information to solve the problem.

As indicated in the book on Page 194, you should attack these problems by drawing a diagram, if relevant, and writing formulas describing what is going on. Once you have the right relational statement, you differentiate with respect to time, and solve. Make sure your answer makes sense and answers the question.

EXAMPLE: A ladder 41 feet long is leaning against a vertical wall. The ladder begins to slip. The ladder's top slides down the wall while its bottom moves along the level ground at a constant speed of 10 ft/sec. How fast is the top of the ladder moving when it is 9 ft above the ground?

SOLUTION: Let x be the distance of the foot of the ladder from the wall and y be the distance of the top from the ground. Both are functions of time t. We are given that $x^2 + y^2 = 41^2$. It follows that $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$, so $\frac{dy}{dt} = \frac{-x}{y} \cdot \frac{dx}{dt}$. At the time in question, $y = 9$, $x = \sqrt{41^2 - 9^2} = 40$, and $\frac{dx}{dt} = +10$. Therefore, $\frac{dy}{dt} = -\frac{400}{9}$ feet/second. The minus is because the distance of the top of the ladder from the ground is decreasing. Notice we do not put in the value $y = 9$ until after we have differentiated.

We will next use the fact that $D_x \sin x = \cos x$, $D_x \cos x = -\sin x$, so

$$D_x \tan x = D_x \frac{\sin x}{\cos x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x = 1 + \tan^2 x.$$

Example: A light beam on a 100 ft tower rotates in a vertical circle at the rate of one revolution per second. Find the speed of the light moving along the ground at a point 1000 feet from the base of the tower.

SOLUTION: Let θ be the angle made by the light with the tower. Let t be time, and x be the distance from the light on the ground to the foot of the tower. Assume that we are looking at the tower so that the rotation is $\frac{d\theta}{dt} = 2\pi$ radians per second. (If we reverse the sign, we reverse the sign of the velocity of the light on the ground, but the speed is still the same.) It now follows that

$$\frac{x}{100} = \tan\theta, \quad \text{or} \quad x = 100\ \tan\theta \text{ feet.}$$

We want to find $\frac{dx}{dt}$ when $x = 1000$ feet.

$$\frac{dx}{dt} = 100\ \sec^2\theta\ \frac{d\theta}{dt} = 200\pi\ \sec^2\theta(t).$$

When $x = 1000$,

$$\sec^2\theta = 1 + \tan^2\theta = 1 + \left(\frac{1000}{100}\right)^2 = 101.$$

Therefore,

$$\frac{dx}{dt} = 20200\pi \text{ feet/second.}$$