1. Dropping or Dropping Down the Course

For students who are having difficulty in this course:

Students may drop down to Math 016 (no credit toward graduation but hours count for full-time student status) until Tuesday, October 10th . Students must pick up a form in 313 Altgeld Hall and have it signed by the Math 016 instructor and return it to 313 Altgeld Hall by 2pm on Oct. 10th. THIS IS A FIRM DEADLINE - NO EXCEPTIONS WILL BE MADE.

If the students have questions, they may see Dianna Armstrong or Alison Champion in 313 AH. Students still have the option to drop the course with no penalty until Friday, Oct. 13.

2. A Tutoring Room is Open

7–9 p.m, Monday, Tuesday, Wednesday, Thursday, Room 140 Lincoln Hall.

3. Homework 11 due Tuesday, October 3 at 9 a.m.

Section 3.8: #12, 14, 16, 22, 30, 36, 40, 42, 50, 52, 56, 60. Put in Parentheses; for example $\cos(\ln(5x))$ not $\cos(\ln 5x)$.

4. Homework 12 due Thursday, October 5 at 9 a.m.

Section 3.9: #18, 22, 24, 36, 38, 42, 46, 52, 56, 58.

5. Written Problem for next week

You have a canal that makes a right angle turn. The width of the canal for the incoming leg is a, and the width of the canal for the outgoing leg is b. What is the longest narrow barge that can be moved around the turn?

Hint: Let θ be the angle formed by a line segment touching the inside corner and terminating at the outside walls of the canal; here the angel is formed by the line segment and the outside wall of the leg of width b. The length of the line segment is

$$L(\theta) = a\sec\theta + b\csc\theta.$$

As θ approaches 0 or $\frac{\pi}{2}$, the length of this line segment approaches $+\infty$. The maximum length of a barge that can make the turn is the minimum value of $L(\theta)$.

6. Exam, Friday October 6, 11 a.m.

On material through related rates (homework for Thursday).

Section 4 (Liu Qi), Section 5 (Liu Qi) Section 6 (Michael Barrus), Section 8 (Scott Weaver) will take the exam in Room 314 Altgeld Hall.

Section 2 (Isaac Goldbring), Section 7 (Isaac Goldbring), Section 9 (Timothy LeSaulnier) will take the exam in Room 100 MSEB (Materials Science Engineering

Building, North-West corner of Green and Mathews.) People in these sections **must** go to this room and not Altgeld Hall to take the exam.

Everyone should by now know their discussion section and section instructor. You will need to enter that on your examination. Bring your U of I identity card to show when turning in the exam.

Review Thursday September 14, Rooms 245, 443, 445 Altgeld Hall, 7-9 p.m.

7. Right triangles

Remember to use radians when working with calculus and trig. functions. We discuss two right triangles, one with $\pi/4$, $\pi/4$ angles and the other with $\pi/6$, $\pi/3$ angles. Also we consider how to find all trig functions of a triangle when you know one of them; draw the correct right triangle. A sample problem that will be on this exam was given in this class. **Come to Class.**

8. More related rates of change

As indicated in the book on Page 194, you should attack these problems by drawing a diagram, if relevant, and writing formulas describing what is going on. Once you have the right relational statement, you differentiate with respect to time, and solve. Make sure your answer makes sense and answers the question.

EXAMPLE: A light beam on a 100 ft tower rotates in a vertical circle at the rate of one revolution per second. Find the speed of the light moving along the ground at a point 1000 feet from the base of the tower.

SOLUTION: Let θ be the angle made by the light with the tower. Let t be time, and x be the distance from the light on the ground to the foot of the tower. Assume that we are looking at the tower so that the rotation is $\frac{d\theta}{dt} = 2\pi$ radians per second. (If we reverse the sign, we reverse the sign of the velocity of the light on the ground, but the speed is still the same.) It now follows that

$$\frac{x}{100} = \tan\theta, \text{ or } x = 100 \tan\theta \text{ feet.}$$

We want to find $\frac{dx}{dt}$ when $x = 1000$ feet.

$$\frac{dx}{dt} = 100 \sec^2\theta \frac{d\theta}{dt} = 200\pi \sec^2\theta(t).$$

When $x = 1000$, $\tan\theta = \frac{1000}{100}$, so

$$\sec^2\theta = 1 + \tan^2\theta = 1 + \left(\frac{1000}{100}\right)^2 = 101.$$

Therefore,

$$\frac{dx}{dt} = 20200\pi \text{ feet/second.}$$

9. Newton's Method

You can find the square root of a positive number a by solving the equation $x^2 - a = 0$. That is, you find the points where the curve $y = x^2 - a$ crosses the x-axis. Given this curve or a more complicated curve $y = f(x)$, you are often interested in points where the curve crosses the x-axis. It is usually easier to find points where tangent lines to the curve cross the x-axis. If you are close to the actual crossing point, you get the answer by approximation. A way to carry this out is Newton's method. It is also called the Newton-Raphson method. The method says do the following:

Step 1 First, make a "good" guess x_1 for the x-value for the place where the curve crosses the x-axis.

Step 2 Then compute the equation of the tangent line at the point $(x_1, f(x_1))$.

Step 3 Now see where the tangent line given by $y - f(x_1) = f'(x_1)(x - x_1)$ crosses the x-axis. That is, solve the equation $-f(x_1) = f'(x_1)(x - x_1)$.

Step 4 The solution, which we denote by x_2, is your next guess.

Step 5 We repeat the process using x_2 as our guess.

Note that x_2 is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

Letting x_2 be our next guess, we see that the next guess after that is $x_3 = x_2 - f(x_2)/f'(x_2)$. We continue in this way, and will usually get numbers x_n closer and closer to the point where the curve $y = f(x)$ crosses the x-axis. Given the n-th point x_n,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

EXAMPLE. To find the square root of a number a, we let $f(x) = x^2 - a$, so that $f'(x) = 2x$, and given a guess at the n^{th} stage x_n, the next guess is

$$x_{n+1} = x_n - \frac{x_n^2 - a}{2x_n} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

For example, if $a = 2$ then $\sqrt{2} = 1.414213562373$. Setting $x_1 = 1$ and trying to compute this value using Newton's method, we have

$$\begin{aligned}
x_2 &= \frac{1}{2}\left(1 + \frac{2}{1}\right) = \frac{3}{2} = 1.5 \\
x_3 &= \frac{1}{2}\left(\frac{3}{2} + \frac{2}{\frac{3}{2}}\right) = \frac{17}{12} = 1.4166 \\
x_4 &= \frac{1}{2}\left(\frac{17}{12} + \frac{2}{\frac{17}{12}}\right) = \frac{577}{408} = 1.41421568 \\
x_5 &= \frac{1}{2}\left(\frac{577}{408} + \frac{2}{\frac{577}{408}}\right) = \frac{665857}{470832} = 1.414213562374
\end{aligned}$$

You should try this with your own calculator and different values of a. It is even more instructive to use a spread-sheet such as Excel.

I will ask you to remember the procedure outlined by Steps 1-5 above.

Example: If $a = 17$, then $f(x) = x^2 - 17$, and for $x_1 = 5$, we have

$$\begin{aligned}
x_2 &= \frac{1}{2}\left(5 + \frac{17}{5}\right) = 4.2 \\
x_3 &= \frac{1}{2}\left(4.2 + \frac{17}{4.2}\right) = 4.12381 \\
x_4 &= \frac{1}{2}\left(4.12381 + \frac{17}{4.12381}\right) = 4.12311 \\
\text{Note that} \quad (4.12311)^2 - 17 &= .0000\,4
\end{aligned}$$

Example: To find a root of $\sin x - x^2$, i.e., a place there $\sin x = x^2$, our first estimate is $x_1 = 1$ radian. Given the n^{th} guess, the next guess is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\sin x_n - x_n^2}{\cos x_n - 2x_n}.$$

Therefore,

$$\begin{aligned}
x_2 &= 1 - \frac{\sin 1 - 1}{\cos 1 - 2} = .89140, \\
x_3 &= .89140 - \frac{\sin .89140 - (.89140)^2}{\cos .89140 - 2 \cdot .89140} = .87698 \\
x_4 &= .87698 - \frac{\sin .87698 - (.87698)^2}{\cos .87698 - 2 \cdot .87698} = .87673 \\
\text{Note that} \quad \sin .87673 - (.87673)^2 &= -4.2152 \times 10^{-6}
\end{aligned}$$

There are functions and guesses where Newton's method will not work.

Example: We know that $f(x) = x^{1/3}$ has its only root at 0. If in fact we try to use Newton's method here, we see that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{1/3}}{\frac{1}{3}x_n^{-2/3}} = x_n - 3x_n = -2x_n.$$

The sequence of guesses alternates sign and doubles the distance of the previous guess from 0, so Newton's method does not work for this function.

Sample Problem: Given a differentiable function $y = f(x)$, you want to find a value x such that $f(x) = 0$. You make a guess x_1 such that the derivative $f'(x_1) \neq 0$. **a)** Write the formula for the tangent line L at the point $(x_1, f(x_1))$. **b)** Your next guess x_2 is the x-coordinate of the point where that tangent line L intersects what line? **Ans: a)** In terms of the variables x and y for the line, $y - f(x_1) = f'(x_1)(x - x_1)$. **Ans: b)** The x-axis.

10. Differentiation and the error for the limit

What does it mean that a function f has a derivative at x. It means that for $y = f(x)$ and a small nonzero change Δx in x, setting $\Delta y = f(x + \Delta x) - f(x)$ we have $\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x)$. That is, there is a function $E(\Delta x)$ with limit 0 at 0 such that $\frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x) + E(\Delta x)$. Multiplying by Δx, we get

$$\Delta y = f(x + \Delta x) - f(x) = f'(x) \cdot \Delta x + E(\Delta x) \cdot \Delta x.$$

Adding $y = f(x)$, we also have

$$y + \Delta y = f(x + \Delta x) = f(x) + f'(x) \cdot \Delta x + E(\Delta x) \cdot \Delta x.$$

This means that the easy first approximation to the point on the graph at $x + \Delta x$ is the point $(x + \Delta x,\ f(x) + f'(x) \cdot \Delta x)$ on the tangent line. The rest, $E(\Delta x) \cdot \Delta x$ is much smaller, so if Δx is very very small, we can't see the difference between the tangent line and the curve within the interval $(x - \Delta x, x + \Delta x)$. Most of calculus is the exploitation of this fact. We will use both the formula for Δy and the formula for $y + \Delta y$.

EXAMPLE. If $y = f(x) = x^3$, then for small $\Delta x \neq 0$,

$$(x + \Delta x)^3 = y + \Delta y = x^3 + 3x^2 \cdot \Delta x + \left[3x \cdot \Delta x + (\Delta x)^2\right] \cdot \Delta x$$

and

$$\Delta y = 3x^2 \Delta x + \left[3x \cdot \Delta x + (\Delta x)^2\right] \cdot \Delta x,$$

The term $\left[3x \cdot \Delta x + (\Delta x)^2\right] \cdot \Delta x$ becomes much smaller than $3x^2 \cdot \Delta x$ for values of Δx close to 0.

Example: For a square of side x, the Area A is given by $A = x^2$, so

$$\Delta A = 2x\Delta x + (\Delta x)^2 .$$

The error is the area of the little corner piece of side Δx.

Example: For a cube of side x, the volume V is given by $V = x^3$, so

$$\Delta V = 3x^2\Delta x + \left[3x \cdot \Delta x + (\Delta x)^2\right] \cdot \Delta x.$$

The error is the volume of pieces much smaller than the 3 pieces of surface area x^2 and thickness Δx.

Sample Question: For $y = x^3$ and a small change Δx, the change in y Δy is the derivative $3x^2$ multiplied by Δx plus a function $E(\Delta x)$ multiplied Δx where $\lim_{\Delta x \to 0} E(\Delta x) = 0$. What is $E(\Delta x)$ in this case. **Ans:** $E(\Delta x)$ is $3x \cdot \Delta x + (\Delta x)^2$.

11. Differentials

The approximation for the change along the tangent line $f'(x) \cdot \Delta x$ is called the differential of f (or of y) at x and Δx. It is denoted by dy. We also write

$$dy = \frac{dy}{dx} \cdot \Delta x \quad \text{or} \quad dy = \frac{dy}{dx} \cdot dx.$$

It is convenient to replace Δx with dx in this formula, but if x is the independent variable, it still means the actual change in x. On the other hand, if x is a function of another variable, say t, then instead of writing

$$dy = \frac{dy}{dx} \cdot \Delta x = \frac{dy}{dx}\left(\frac{dx}{dt}\Delta t + \text{error} \cdot \Delta t\right)$$

we think of y as a function of t, forget the error, and set

$$dy = \frac{dy}{dx} \cdot dx = \frac{dy}{dx} \cdot \frac{dx}{dt}\Delta t = \frac{dy}{dx} \cdot \frac{dx}{dt}dt.$$

That is, all changes of dependent variables are calculated along the appropriate tangent lines.

Often, dy is called the **linear approximation** to the change in the function because you are using the change along the tangent line dy to approximate the actual change Δy.

Here are some differentials: $d(x^2) = 2xdx$; $d\sin x = \cos x\, dx$; $d\sqrt{x} = \frac{1}{2\sqrt{x}}dx$; etc. Remember, if x is the independent variable, dx just means the change in x. Otherwise, dx is the differential of x. This is short hand for the chain rule.