1. Dropping or Dropping Down the Course

For students who are having difficulty in this course:

Students may drop down to Math 016 (no credit toward graduation but hours count for full-time student status) until Tuesday, October 10th . Students must pick up a form in 313 Altgeld Hall and have it signed by the Math 016 instructor and return it to 313 Altgeld Hall by 2pm on Oct. 10th. THIS IS A FIRM DEADLINE - NO EXCEPTIONS WILL BE MADE.

If the students have questions, they may see Dianna Armstrong or Alison Champion in 313 AH. Students still have the option to drop the course with no penalty until Friday, Oct. 13.

2. Problems with entering homework

If you believe that you have entered a correct answer for the homework that the website will not accept, be sure you have carefully read the problem and the format needed. If you have done this, you should print out the problem with your solution from the screen and hand it in to your TA. If he agrees with you, he will note the need for the addition of a point to the score for that homework, and add that point at the end of the term.

I have asked the publisher to fix the problems with the website that we did not have last year.

3. A Tutoring Room is Open

7–9 p.m, Monday, Tuesday, Wednesday, Thursday, Room 140 Lincoln Hall.

4. Homework 11 due Tuesday, October 3 at 9 a.m.

Section 3.8: #12, 14, 16, 22, 30, 36, 40, 42, 50, 52, 56, 60. Put in Parentheses; for example $\cos(\ln(5x))$ not $\cos(\ln 5x)$.

5. Homework 12 due Thursday, October 5 at 9 a.m.

Section 3.9: #18, 22, 24, 36, 38, 42, 46, 52, 56, 58.

6. Written Problem for this week

You have a canal that makes a right angle turn. The width of the canal for the incoming leg is a, and the width of the canal for the outgoing leg is b. What is the longest narrow barge that can be moved around the turn?

Hint: Let θ be the angle formed by a line segment touching the inside corner and terminating at the outside walls of the canal; here the angel is formed by the line segment and the outside wall of the leg of width b. The length of the line segment is

$$L(\theta) = a \sec\theta + b \csc\theta.$$

As θ approaches 0 or $\frac{\pi}{2}$, the length of this line segment approaches $+\infty$. The maximum length of a barge that can make the turn is the minimum value of $L(\theta)$.

7. Exam, Friday October 6, 11 a.m.

On material through related rates (homework for Thursday).

Section 4 (Liu Qi), Section 5 (Liu Qi) Section 6 (Michael Barrus), Section 8 (Scott Weaver) will take the exam in Room 314 Altgeld Hall.

Section 2 (Isaac Goldbring), Section 7 (Isaac Goldbring), Section 9 (Timothy LeSaulnier) will take the exam in Room 100 MSEB (Materials Science Engineering Building, North-West corner of Green and Mathews.) People in these sections **must** go to this room and not Altgeld Hall to take the exam.

Everyone should by now know their discussion section and section instructor. You will need to enter that on your examination. Bring your U of I identity card to show when turning in the exam.

Review Thursday September 14, Rooms 245, 443, 445 Altgeld Hall, 7-9 p.m.

8. Right triangles

Remember to use radians when working with calculus and trig. functions. In the last class we discussed two right triangles, one with $\pi/4$, $\pi/4$ angles and the other with $\pi/6$, $\pi/3$ angles. Also we considered how to find all trig functions of a triangle when you know one of them: draw the correct right triangle. A sample of a problem that will be on this exam was given in the last class. **Come to Class.**

9. Sample Problem for Newton's Method

Given a differentiable function $y = f(x)$, you want to find a value x such that $f(x) = 0$. You make a guess x_1 such that the derivative $f'(x_1) \neq 0$. **a)** Write the formula for the tangent line L at the point $(x_1, f(x_1))$. **b)** Your next guess x_2 is the x-coordinate of the point where that tangent line L intersects what line? **Ans: a)** In terms of the variables x and y for the line, $y - f(x_1) = f'(x_1)(x - x_1)$. **Ans: b)** The x-axis.

10. Differentials

We have said that for a differentiable function $y = f(x)$, at a point x

$$\Delta y = f(x + \Delta x) - f(x) = f'(x) \cdot \Delta x + E(\Delta x) \cdot \Delta x,$$

where $\lim_{\Delta x \to 0} E(\Delta x) = 0$. Adding $y = f(x)$, we also have

$$y + \Delta y = f(x + \Delta x) = f(x) + f'(x) \cdot \Delta x + E(\Delta x) \cdot \Delta x.$$

This means that the easy first approximation to the point on the graph at $x + \Delta x$ is the point $(x + \Delta x,\ f(x) + f'(x) \cdot \Delta x)$ on the tangent line.

The approximation for the change along the tangent line $f'(x) \cdot \Delta x$ is called the differential of f (or of $y = f(x)$) at x and Δx. It is denoted by dy. We also write

$$dy = \frac{dy}{dx} \cdot \Delta x \quad \text{or} \quad dy = \frac{dy}{dx} \cdot dx.$$

It is convenient to replace Δx with dx in this formula, but if x is the independent variable, it still means the actual change in x. On the other hand, if x is a function of another variable, say t, then instead of writing

$$dy = \frac{dy}{dx} \cdot \Delta x = \frac{dy}{dx}\left(\frac{dx}{dt}\Delta t + \text{error} \cdot \Delta t\right)$$

we think of y as a function of t, we forget the error, and we set

$$dy = \frac{dy}{dx} \cdot dx = \frac{dy}{dx} \cdot \frac{dx}{dt}\Delta t = \frac{dy}{dx} \cdot \frac{dx}{dt}dt.$$

That is, all changes of dependent variables are calculated along the appropriate tangent lines.

Often, dy is called the **linear approximation** to the change in the function because you are using the change along the tangent line dy to approximate the actual change of the output Δy.

Here are some differentials: $d(x^2) = 2x dx$; $d \sin x = \cos x \, dx$; $d\sqrt{x} = \frac{1}{2\sqrt{x}}dx$; etc. Remember, if x is the independent variable, dx just means the change in x. Otherwise, dx is the differential of x. This is short hand for the chain rule. Do not confuse the derivative with the differential.

You have problems for homework where you approximate actual changes with differentials. For $y = f(x)$, we approximate $f(x + \Delta x) = y + \Delta y$ with $y + dy = f(x) + f'(x)dx$.

EXAMPLE: $10^3 = 1000$. If we want the value of 10.1^3 we can instead evaluate the differential for $y = x^3$ at $x = 10$ and $\Delta x = .1$. This is $3 \cdot 10^2 \cdot 1/10 = 30$, so our approximation is $10^3 + dy = 1030$. The actual value is $10.1^3 = 1030.301$.

EXAMPLE: The Earth is approximately a sphere of radius $r = 4000$ miles, and the ice at the poles is $8,000,000$ miles^3 in volume. We want to find the increased radius of the earth that would result if the ice all melted and this volume were uniformly added to the Earth's volume. Now $V = \frac{4}{3}\pi r^3$, and $dV = 4\pi r^2 dr$. We estimate the answer by finding the value for dr if we take $r = 4000$ and $dV = 8,000,000$.

$$dr = \frac{dV}{4\pi r^2} = \frac{8,000,000}{4\pi(4000)^2} = \frac{1}{8\pi} = .0398 \text{ miles.}$$

In feet, this is $.0398 \cdot 5280 = 210$ feet.

Sample Problem Use differentials, that is, a linear approximation, to estimate $30^{1/5}$. We know that $2^5 = 32$, whence $32^{1/5} = 2$. Let $y = x^{1/5}$. Then $dy = \frac{1}{5}x^{-4/5}dx$. Working at the point $x = 32$, $dx = -2$ with the formula $y + \Delta y \approx y + dy = y + \frac{dy}{dx}dx$, we have

$$30^{\frac{1}{5}} \approx 32^{\frac{1}{5}} + \frac{1}{5}(32)^{\frac{-4}{5}}(-2) = 1.975.$$

The actual value is 1.9744.

Example: Use the fact that $2^\circ = \frac{\pi}{180} \cdot 2 = \frac{\pi}{90}$ and differentials to find

$$\cos 62^\circ = \cos(60^\circ + 2^\circ) = \cos\left(\frac{\pi}{3} + \frac{\pi}{90}\right).$$

For $y = \cos x$, $dy = -(\sin x)\, dx$, For $x = \frac{\pi}{3}$ and $dx = \frac{\pi}{90}$, we have

$$\cos\left(\frac{\pi}{3} + \frac{\pi}{90}\right) \approx \cos\left(\frac{\pi}{3}\right) + \cos'\left(\frac{\pi}{3}\right) \cdot \frac{\pi}{90} = \frac{1}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\pi}{90} = 0.46977.$$

The actual value is $\cos\left(\frac{\pi}{3} + \frac{\pi}{90}\right) = 0.46947$.

Example: A hemispherical dome with radius 50 feet is given a coat of paint .01 inches thick. We want to approximate the increase in the volume of the dome to estimate the amount of paint needed. Now for half a sphere, $V = \frac{2}{3}\pi r^3$, and $dV = 2\pi r^2 dr$. In this case, $r = 50 \cdot 12$ in. and $dr = \frac{1}{100}$ in., so

$$\begin{aligned} dV &\approx \frac{2\pi \cdot (50 \cdot 12)^2}{100} = 2\pi \cdot 25 \cdot 144 \text{ cubic inches} \\ &= \frac{2\pi \cdot 25 \cdot 144}{231} \text{gallons} \approx 98 \text{ gallons.} \end{aligned}$$

11. MEAN VALUE THEOREM

Here is one of the most useful facts about the derivative:

Theorem 1 [Mean Value Theorem]**.** *Let $y = f(x)$ be continuous on a closed interval $[a, b]$ and have a derivative everywhere on the open interval (a, b). Then there is a point c in (a, b) such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

This theorem says that the slope of the secant line joining the points $(a, f(a))$ and $(b, f(b))$ equals the slope of the tangent line at some point c in (a, b). This is some point, not every point, but some point.

A practical example is that if you drive a car for 4 hours and travel 280 miles, so that your average speed is 70 miles per hour, then at some time during your trip, your speedometer must show an instantaneous speed to 70 mph. It does not mean that you always are going 70 mph!

Here is a special case of the Mean Value Theorem.

SPECIAL CASE (ROLLE'S THEOREM). Given f as above, if $f(a) = f(b) = 0$, then for some c in (a, b), $f'(c) = 0$.

The proof of this special case consists of noting that either f is identically equal to 0 on the interval, in which case f' is identically 0, or because f is continuous on the interval, it must takes a maximum or minimum value at some point c inside the interval. We have seen that in the latter case, $f'(c) = 0$.

Sample Problem: Suppose $y = f(x)$ is a differentiable function that is not always 0 on the interval $[2, 7]$ but $f(2) = f(7) = 0$. Must the derivative f' take the value 0 at some point between 2 and 7? Why? Ans: Yes, because f is continuous, $f(x)$ must either take a positive maximum value or a negative minimum value at some point between 2 and 7. at such a point c, $f'(c) = 0$.

PROOF OF MEAN VALUE THEOREM (General case). We will only sketch this proof in class. Let m be the slope of the secant line joining the points $(a, f(a))$ and $(b, f(b))$; i.e.,

$$m = \frac{f(b) - f(a)}{b - a}.$$

Using the variable y_s for the secant line, it is given by the formula

$$y_s(x) = f(a) + m(x - a).$$

For any x in $[a, b]$, let $y = f(x)$ denote the y value on the curve, and let $\phi(x)$ denote the difference between the graph of f and the secant line. That is, for each x in $[a, b]$,

$$\phi(x) := y - y_s = f(x) - f(a) - m(x - a).$$

Then $\phi(a) = \phi(b) = 0$, and so by Rolle's Theorem there is a point c in (a, b) with

$$0 = \phi'(c) = f'(c) - m.$$

That is, $f'(c) = m$. □

Corollary 2. *If f is continuous on $[a, b]$ and $f'(x) = 0$ for all x in (a, b), then $f(x)$ is identically equal to the constant value $f(a)$ for all x in $[a, b]$.*

Proof. The Mean Value Theorem tells us that the slope

$$\frac{f(x) - f(a)}{x - a} = 0$$

for each x with $a < x \leq b$. Therefore, $f(x) - f(a) = 0$ for all x in the interval.

Corollary 3. *If f and g are two functions continuous on $[a, b]$ and the derivatives of f and g exist and are equal at all points of (a, b), i.e.,*

$$f'(x) = g'(x) \quad \textit{for all } x \textit{ in } (a, b),$$

then for some constant C,

$$f(x) = g(x) + C \qquad \textit{for all } x \textit{ in } (a, b).$$

Proof. Let $C = f(a) - g(a)$. Since $f - g$ is continuous on $[a, b]$ and $f' - g' = 0$ on (a, b), for each x in (a, b)

$$f(x) - g(x) = f(a) - g(a) = C.$$

Remark 4. *We have seen that for $\Delta x \neq 0$, if we go from x to $x + \Delta x$, then for some function E of Δx with limit 0 at 0,*

$$\Delta y = f'(x) \cdot \Delta x + E(\Delta x) \cdot \Delta x.$$

Now we see that for some point c between x and $x + \Delta x$, $\Delta y = f'(c) \cdot \Delta x$. That is, if we evaluate f' at the unknown point c, then we can forget the error $E(\Delta x) \cdot \Delta x$.

Sample Problem: Suppose f is a differentiable function on the real line and the derivative $f'(x) \geq 9$ for all x. Is it possible that $f(1) = 5$ and $f(2) = 8$? Explain.
Ans. The answer is no, $(f(2) - f(1))/(2 - 1) = 3$, but there is no point c between 1 and 2 where $f'(c) = 3$.

Theorem 5. *Assume f is a continuous function on an interval I. If $f'(x) > 0$ everywhere on I except perhaps at the endpoints, then f is* ***strictly increasing*** *on that interval. That is, if $x_1 < x_2$, then $f(x_1) < f(x_2)$. On the other hand, if $f'(x) < 0$ everywhere on I except perhaps at the endpoints, then f is* ***strictly decreasing*** *on the interval. That is, if $x_1 < x_2$, then $f(x_1) > f(x_2)$.*

(NOTE: your book uses the terms *increasing* and *decreasing*, but this usually means a non-strict inequality $\leq$ or $\geq$ holds.)
Proof. Assume $f'(x) > 0$ at all points except perhaps the endpoints of I, and fix two points $x_1 < x_2$ in I. Then for some point c with $x_1 < c < x_2$,

$$\Delta y = f(x_2) - f(x_1) = f'(c) \cdot \Delta x = f'(c)(x_2 - x_1) > 0, \quad \text{that is,} \quad f(x_2) > f(x_1).$$

Now assume $f'(x) < 0$ at all points except perhaps the endpoints of I, and fix two points $x_1 < x_2$ in I. Then for some point c with $x_1 < c < x_2$,

$$\Delta y = f(x_2) - f(x_1) = f'(c) \cdot \Delta x = f'(c)(x_2 - x_1) < 0, \text{ that is, } f(x_2) < f(x_1).$$