1. Dropping or Dropping Down the Course

For students who are having difficulty in this course:

Students may drop down to Math 016 (no credit toward graduation but hours count for full-time student status) until Tuesday, October 10th . Students must pick up a form in 313 Altgeld Hall and have it signed by the Math 016 instructor and return it to 313 Altgeld Hall by 2pm on Oct. 10th. THIS IS A FIRM DEADLINE - NO EXCEPTIONS WILL BE MADE.

If the students have questions, they may see Dianna Armstrong or Alison Champion in 313 AH. Students still have the option to drop the course with no penalty until Friday, Oct. 13.

2. Problems with entering homework

If you believe that you have entered a correct answer for the homework that the website will not accept, be sure you have carefully read the problem and the format needed. If you have done this, you should print out the problem with your solution from the screen and hand it in to your TA. If he agrees with you, he will note the need for the addition of a point to the score for that homework, and add that point at the end of the term.

I have asked the publisher to fix the problems with the website that we did not have last year.

3. A Tutoring Room is Open

7–9 p.m, Monday, Tuesday, Wednesday, Thursday, Room 140 Lincoln Hall.

4. Homework 12 due Thursday, October 5 at 9 a.m.

Section 3.9: #18, 22, 24, 36, 38, 42, 46, 52, 56, 58.

5. Homework 13 due Tuesday, October 10 at 9 a.m.

Section 3.9: #16, 28, 54 (Give an exact answer, not a decimal).

Section 3.10: #2, 6, 18, 20, 44.

6. Higher Derivatives

You will be expected to know the notation for evaluating higher derivatives at a point. This is written out at the end of these notes.

7. Written problem for next week

Suppose f is a function on the real line such that $|f'(x)| \leq 1$ for all real numbers x. Use the Mean Value Theorem to show that for any two numbers x_1 and x_2, we must have

$$|f(x_1) - f(x_2)| \leq |x_1 - x_2| .$$

Hint: You will also use the fact that for all numbers a and b with $b \neq 0$ we have $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$.

8. Solution to written problem for this week

You have a canal that makes a right angle turn. The width of the canal for the incoming leg is a, and the width of the canal for the outgoing leg is b. What is the longest narrow barge that can be moved around the turn?

Ans: Let θ be the angle formed by a line segment touching the inside corner and terminating at the outside walls of the canal; here the angel is formed by the line segment and the outside wall of the leg of width b. The length of the line segment is

$$L(\theta) = a\sec\theta + b\csc\theta.$$

As θ approaches 0 or $\frac{\pi}{2}$, the length of this line segment approaches $+\infty$. Therefore, the maximum length of a barge that can make the turn is the minimum value of $L(\theta)$, which is at a value θ where $L'(\theta) = 0$. Now we make the following calculations to find the desired value of θ:

$$\begin{aligned} L'(\theta) &= a\sec\theta\tan\theta - b\csc\theta\cot\theta = 0 \\ a\sec\theta\tan\theta &= b\csc\theta\cot\theta \\ \frac{\sec\theta\tan\theta}{\csc\theta\cot\theta} &= \frac{\sin\theta}{\cos\theta}\cdot\tan^2\theta = \tan^3\theta = \frac{b}{a}, \end{aligned}$$

or $\tan\theta = \frac{b^{1/3}}{a^{1/3}}$. Now we consider a right triangle with the side adjacent to the angle θ having length $a^{1/3}$, while the side opposite θ has length $b^{1/3}$. This gives the right value for $\tan\theta$, and the hypotenuse has length $\sqrt{a^{2/3}+b^{2/3}}$. Using this triangle, we see that the length

$$\begin{aligned} L(\theta) &= a\sec\theta + b\csc\theta \\ &= a\frac{\sqrt{a^{2/3}+b^{2/3}}}{a^{1/3}} + b\frac{\sqrt{a^{2/3}+b^{2/3}}}{b^{1/3}} \\ &= \left(a^{2/3}+b^{2/3}\right)\sqrt{a^{2/3}+b^{2/3}} = \left(a^{2/3}+b^{2/3}\right)^{3/2}. \end{aligned}$$

9. Exam, Friday October 6, 11 a.m.

On material through related rates (homework for Thursday).

Section 4 (Liu Qi), Section 5 (Liu Qi) Section 6 (Michael Barrus), Section 8 (Scott Weaver) will take the exam in Room 314 Altgeld Hall.

Section 2 (Isaac Goldbring), Section 7 (Isaac Goldbring), Section 9 (Timothy LeSaulnier) will take the exam in Room 100 MSEB (Materials Science Engineering Building, North-West corner of Green and Mathews.) People in these sections **must** go to this room and not Altgeld Hall to take the exam.

Everyone should by now know their discussion section and section instructor. You will need to enter that on your examination. Bring your U of I identity card to show when turning in the exam.

Review Thursday September 14, Rooms 245, 443, 445 Altgeld Hall, 7-9 p.m.

10. Sample Problem for this coming exam

Let $y = \ln x$ for $x > 0$. **Use implict differentiation** together with the equation $x = e^y$ and the rule for differentiating the exponential function to derive the formula for the derivative dy/dx.

11. More on Mean Value Theorem

Here is one of the most useful facts about the derivative:

Theorem 1 [Mean Value Theorem]**.** *Let $y = f(x)$ be continuous on a closed interval $[a, b]$ and have a derivative everywhere on the open interval (a, b). Then there is a point c in (a, b) such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

This theorem says that the slope of the secant line joining the points $(a, f(a))$ and $(b, f(b))$ equals the slope of the tangent line at some point c in (a, b). This is at some point, not every point, but some point.

Corollary 2. *If f is continuous on $[a, b]$ and $f'(x) = 0$ for all x in (a, b), then $f(x)$ is identically equal to the constant value $f(a)$ for all x in $[a, b]$.*

Proof. The Mean Value Theorem tells us that the slope

$$\frac{f(x) - f(a)}{x - a} = 0$$

for each x with $a < x \leq b$. Therefore, $f(x) - f(a) = 0$ for all x in the interval.

Corollary 3. *If f and g are two functions continuous on $[a, b]$ and the derivatives of f and g exist and are equal at all points of (a, b), i.e.,*

$$f'(x) = g'(x) \text{ for all } x \text{ in } (a, b),$$

then for some constant C,

$$f(x) = g(x) + C \quad \text{for all } x \text{ in } (a, b).$$

Proof. Let $C = f(a) - g(a)$. Since $f - g$ is continuous on $[a, b]$ and $f' - g' = 0$ on (a, b), for each x in (a, b)

$$f(x) - g(x) = f(a) - g(a) = C.$$

Remark 4. *We have seen that for* $\Delta x \neq 0$*, if we go from* x *to* $x + \Delta x$*, then for some function* E *of* Δx *with limit* 0 *at* 0*,*

$$\Delta y = f'(x) \cdot \Delta x + E(\Delta x) \cdot \Delta x.$$

Now we see that for some point c *between* x *and* $x + \Delta x$*,* $\Delta y = f'(c) \cdot \Delta x$*. That is, if we evaluate* f' *at the unknown point* c*, then we can forget the error* $E(\Delta x) \cdot \Delta x$*.*

Sample Problem: Suppose f is a differentiable function on the real line and the derivative $f'(x) \geq 9$ for all x. Is it possible that $f(1) = 5$ and $f(2) = 8$? Explain.
Ans. The answer is no, $(f(2) - f(1))/(2 - 1) = 3$, but there is no point c between 1 and 2 where $f'(c) = 3$.

Theorem 5. *Assume* f *is a continuous function on an interval* I*. If* $f'(x) > 0$ *everywhere on* I *except perhaps at the endpoints, then* f *is* ***strictly increasing*** *on that interval. That is, if* $x_1 < x_2$*, then* $f(x_1) < f(x_2)$*. On the other hand, if* $f'(x) < 0$ *everywhere on* I *except perhaps at the endpoints, then* f *is* ***strictly decreasing*** *on the interval. That is, if* $x_1 < x_2$*, then* $f(x_1) > f(x_2)$*.*

(NOTE: your book uses the terms *increasing* and *decreasing*, but this usually means a non-strict inequality $\leq$ or $\geq$ holds.)
Proof. Assume $f'(x) > 0$ at all points except perhaps the endpoints of I, and fix two points $x_1 < x_2$ in I. Then for some point c with $x_1 < c < x_2$,

$$\Delta y = f(x_2) - f(x_1) = f'(c) \cdot \Delta x = f'(c)(x_2 - x_1) > 0, \quad \text{that is,} \quad f(x_2) > f(x_1).$$

Now assume $f'(x) < 0$ at all points except perhaps the endpoints of I, and fix two points $x_1 < x_2$ in I. Then for some point c with $x_1 < c < x_2$,

$$\Delta y = f(x_2) - f(x_1) = f'(c) \cdot \Delta x = f'(c)(x_2 - x_1) < 0, \text{ that is, } f(x_2) < f(x_1).$$

Remark 6. If a function has a **continuous** derivative, then the derivative can not change from positive to negative or negative to positive without going through the value 0. Therefore, once you know where the derivative is 0, you only need to check at one point between each pair of zeros of the derivative to know what the sign of the derivative is everywhere between those points. On the other hand, if the derivative is a polynomial that you can factor, then you can check the sign of each factor to determine the sign of the derivative.

Example: Consider the function $f(x) = x^3 - 12x + 1$. Where is it increasing? Where is it decreasing? Now $f'(x) = 3x^2 - 12 = 0$ at $x = -2$ and $x = 2$. Because f' is continuous, these are the only places that it can change sign. That is, the derivative can not change from negative to positive or positive to negative without going through 0. For values of $x < -2$, $f'(x) > 0$, so the function itself is increasing in this interval. Between -2 and 2, the sign of the derivative is the same as at 0, i.e., negative. The function is decreasing in this interval. For values of $x > 2$, the derivative is positive, so the function is increasing.

On the other hand, we can factor the derivative to get

$$f'(x) = 3x^2 - 12 = 3(x-2)(x+2).$$

Ignoring the factor 3, for values of x below -2, both factors are negative, so the derivative is positive. For values of x between -2 and 2, only one factor is negative, so the derivative is negative. For values of x larger than 2, both factors are positive, so the derivative is positive.

Now notice that $f(0) = 1$. It can not again equal 1 in the interval $[0, 1]$, because if it did, by the Mean Value Theorem the derivative would have to equal 0 somewhere in the interval $(0, 1)$, and it does not equal 0 anywhere in $(0, 1)$.

Sample Problem: Consider the equation $f(x) = x^3 + 9x^2 + 33x - 8 = 0$; show that it has exactly one real solution. You may use Rolle's Theorem and the fact that $f(-1) < 0$, and $f(1) > 0$. **Ans:** Now $f(-1) < 0$, and $f(1) > 0$, so by the Intermediate Value Theorem, the graph of the function must cross the x axis between -1 and 1. On the other hand, $f'(x) = 3x^2 + 18x + 33 = 0$ when $x^2 + 6x + 11 = 0$. There are no real solutions for the last equation. Therefore, by Rolle's theorem, there can only be one point where $f(x) = 0$.

Note: In general, a function with a derivative that exists but is never zero on an open interval (a, b) can have at most one zero in (a, b). To see this, assume there were two places $x_1 < x_2$ in (a, b) with $f(x_1) = 0$ and $f(x_2) = 0$, then by Rolle's Theorem, there would have to be a point c in (x_1, x_2) where $f'(c) = 0$. However, no such point exists.

12. HIGHER DERIVATIVES

We can also get information for the graph of a function by looking at the derivative of the derivative and noting where it is 0, positive, and negative.

Once we have the derivative f' of a function, we can ask if the derivative of the derivative exists. For $y = f(x)$, we write the derivative of the derivative at x as

$$f''(x) = D_x^2 y = y'' = \frac{d^2y}{dx^2}.$$

The last notation comes from the idea that we are applying the "operation" $\frac{d}{dx}$ to the derivative $\frac{dy}{dx}$. The notation D_x^2 is similarly inspired. If we take n derivatives, we write

$$f^{(n)}(x) = D_x^n y = y^{(n)} = \frac{d^n y}{dx^n}.$$

EXAMPLE: For $y = f(x) = x^3 + x^2 - 6x$,

$$f'(x) = \frac{dy}{dx} = 3x^2 + 2x - 6,$$

$$f''(x) = \frac{d^2y}{dx^2} = 6x + 2, \text{ and } \quad f^{(3)}(x) = \frac{d^3y}{dx^3} = 6.$$

When we want to indicate the value of a derivative at a particular point of the domain, and we are using the Leibniz d/dx notation, we do it as indicated in the following example: If

$$y = f(x), \text{ then } f'(a) = \frac{dy}{dx}|_{x=a}.$$

For example, if $y = x^3$, then

$$\frac{dy}{dx} = 3x^2, \ \frac{d^2y}{dx^2} = 6x, \ \frac{dy}{dx}|_{x=5} = 75, \ \frac{d^2y}{dx^2}|_{x=5} = 30.$$

Note this notation for evaluating higher derivatives at a point.