## 1. Dropping or Dropping Down the Course

For students who are having difficulty in this course:

Students may drop down to Math 016 (no credit toward graduation but hours count for full-time student status) until Tuesday, October 10th . Students must pick up a form in 313 Altgeld Hall and have it signed by the Math 016 instructor and return it to 313 Altgeld Hall by 2pm on Oct. 10th. THIS IS A FIRM DEADLINE - NO EXCEPTIONS WILL BE MADE.

If the students have questions, they may see Dianna Armstrong or Alison Champion in 313 AH. Students still have the option to drop the course with no penalty until Friday, Oct. 13.

## 2. Problems with entering homework

If you believe that you have entered a correct answer for the homework that the website will not accept, be sure you have carefully read the problem and the format needed. If you have done this, you should print out the problem with your solution from the screen and hand it in to your TA. If he agrees with you, he will note the need for the addition of a point to the score for that homework, and add that point at the end of the term.

I have asked the publisher to fix the problems with the website that we did not have last year.

## 3. A Tutoring Room is Open

7–9 p.m, Monday, Tuesday, Wednesday, Thursday, Room 140 Lincoln Hall.

## 4. Homework 13 due Tuesday, October 10 at 9 a.m.

Section 3.9: #16, 28, 54 (Give an exact answer, not a decimal).

Section 3.10: #2, 6, 18, 20, 44.

## 5. Homework 14 due Thursday, October 12 at 9 a.m.

Section 4.2: #6, 10, 14, 26, 32, 40, 42, 48.

Section 4.3: #2, 4.

## 6. Written problem for this week

Suppose $f$ is a function on the real line such that $|f'(x)| \le 1$ for all real numbers $x$. Use the Mean Value Theorem to show that for any two numbers $x_1$ and $x_2$, we must have

$$|f(x_1) - f(x_2)| \le |x_1 - x_2| .$$

Hint: You will also use the fact that for all numbers $a$ and $b$ with $b \ne 0$ we have $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$.

## 7. Grade on Exam 2

$\leq 34$, F, 16. $[35, 39]$, D-, 11. $[40, 44]$, D, 9. $[45, 49]$, D+, 8. $[50, 54]$, C-, 11. $[55, 59]$, C, 19. $[60, 64]$, C+, 26. $[65, 69]$, B-, 29. $[70, 74]$, B, 14. $[75, 79]$, B+, 16. $[80, 87]$, A-, 11. $[88, 93]$, A, 8. $[94, 100]$, A+, 6.

## 8. Answers to Exam 2

**1)** Find the following derivatives. Show all your work and **do not simplify**.

**a)** $D_x\left(\frac{3x^2-5x+2}{4+\sin x}\right) = \frac{(6x-5)(4+\sin x)-(3x^2-5x+2)(\cos x)}{(4+\sin x)^2}$.

**1b)** $D_x\left(\ln(x^2+7)\cdot\sec(x^3-3)\right) = \frac{2x}{x^2+7}\cdot\sec(x^3-3)+\ln(x^2+7)\cdot\sec(x^3-3)\cdot\tan(x^3-3)\cdot 3x^2$.

**1c)** $D_x\left(\cos^5(e^{x^2})\right) = -5\cos^4(e^{x^2})(\sin(e^{x^2}))(e^{x^2})(2x)$.

**2 a)** Write the equation of the tangent line to the graph of the curve given by the equation $\frac{1}{1+y}+\ln x = \frac{x}{2}$ at the point $(1,1)$. **Ans:** Using implicit differentiation, we have

$$\frac{-1}{(1+y)^2}\frac{dy}{dx}+\frac{1}{x}=\frac{1}{2}.$$

At the point $(1,1)$ this is $(-1/4)\,dy/dx = -1/2$, or $dy/dx = 2$. The equation of the tangent line is $y-1 = 2(x-1)$, or $y = 2x-1$.

**2 b)** The position function of a particle moving in a horizontal straight line with motion to the right being positive is given by $x(t) = -2t^3-3t^2+12t+5$ feet for all times $t \geq 0$, where the time $t$ is given in seconds. Find the velocity and the acceleration of the particle at any nonnegative time $t$, and find the particle's position at a positive time when its velocity is zero. **Ans:** The velocity $v(t) = -6t^2-6t+12$ feet/sec, and the acceleration $a(t) = -12t-6$ feet/sec/sec. The velocity $v(t) = 0$ at a positive time when $t = 1$, and at that time $x(1) = 12$ feet.

**3)** Let $f(x) = 2x^3-3x^2+6$ on the interval $[0,2]$. List all critical points of $f$ on this interval, and find the maximum and minimum values of $f(x)$ on this interval. **Ans:** $f'(x) = 6x^2-6x = 0$ when $x = 0$ and $x = 1$. The critical points are 0, 1, and 2. $f(0) = 6$, $f(1) = 5$, and $f(2) = 10$. The minimum value of $f(x)$ on $[0,2]$ is 5 and the maximum value is 10.

**4)** Let $f(x) = x$ and $g(x) = x^5$. Find the value of $x$ in the interval $[0,1]$ for which $f(x)-g(x)$ takes its maximum value, and explain why you chose this critical point for the answer. **Ans:** The derivative of $x-x^5$ is $1-5x^4$, and this is 0 when $x = 5^{-1/4}$. At the other critical points, the end points, the difference is 0.

**5 a)** Let $y = \ln x$ for $x > 0$. **Prove that** $dy/dx = 1/x$ **by applying implicit differentiation** to the equation $x = e^y$. **Show all your work. Ans:** Since $x = e^y$, $1 = e^y(dy/dx)$, or $dy/dx = 1/e^y = 1/x$.

**5 b)** Suppose $0 \le \theta \le \pi/2$ and $\tan\theta = 5/3$. Draw and label the appropriate triangle and use it to find $\sin\theta$ and $\sec\theta$. **Ans:** For the appropriate triangle, the adjacent side is 3, the opposite side is 5, and the hypotenuse is $\sqrt{34}$. Therefore, $\sin\theta = 5/\sqrt{34}$ and $\sec\theta = \sqrt{34}/3$.

**5 c)** For what special value of $a > 1$ is $\lim_{\Delta x\to 0} \frac{a^{\Delta x} - a^0}{\Delta x} = \lim_{\Delta x\to 0} \frac{a^{\Delta x} - 1}{\Delta x} = 1$? **Ans:** $e$.

**6)** A train 16 feet wide is approaching a calm observer standing in the middle of the track. The train is traveling at 32 ft/sec. Let $y$ be the distance from the observer to the middle of the front of the train, and let $z$ be the distance from the observer to the edge of the front of the train at the observer's right. Note that $y^2+8^2 = z^2$. Let $\theta$ be the angle in radians between the line going from the observer to the middle of the front of the train and the line going from the observer to the edge of the front of the train at the observer's right. Find the rate of increase (in radians per second) of the angle $\theta$ when $y = 24$ feet. (Hint: First express $y$ as a constant times a trigonometric function of $\theta$.) **Ans:** Since $y = 8\cot\theta$, and the distance from the observer to the train is decreasing, $dy/dt = -32 = -8\csc^2\theta(d\theta/dt)$, or $d\theta/dt = 4\sin^2\theta$. At the moment in question, $y = 24$, so $\sin\theta = 1/\sqrt{10}$, so $d\theta/dt = 4/10 = 2/5$ radians/sec.

## 9. Examples for Higher Derivatives

When we want to indicate the value of a derivative at a particular point of the domain, and we are using the Leibniz $d/dx$ notation, we do it as indicated in the following example: If

$$y = f(x), \text{ then } f'(a) = \frac{dy}{dx}|_{x=a}.$$

For example, if $y = x^3$, then

$$\frac{dy}{dx} = 3x^2,\ \frac{d^2y}{dx^2} = 6x,\ \frac{dy}{dx}|_{x=5} = 75,\ \frac{d^2y}{dx^2}|_{x=5} = 30.$$

Recall that the first derivative of position with respect to time is the velocity. The second derivative is acceleration.

**Example:** For a freely falling body, with up as the positive $y$ direction, the second derivative of the height $y$ is $d^2y/dt^2 = a = -g$. Using the fact that two functions with the same derivative differ only by a constant, the first derivative must be $-gt + v_0$

where $v_0$ is the initial velocity, and the height $y = \frac{-1}{2}gt^2 + v_0 t + y_0$, where $y_0$ is the initial height.

**Example:** We are to twice differentiate implicitly the relation $4\tan y = x^3$. Differentiating implicitly once, we have

$$4\sec^2 y \frac{dy}{dx} = 3x^2.$$

Differentiating implicitly again, we have

$$8(\sec y)(\sec y \tan y)\left(\frac{dy}{dx}\right)^2 + 4\sec^2 y \frac{d^2y}{dx^2} = 6x.$$

**Example:** We are to twice differentiate implicitly the relation $\sin^2 x + \cos^2 y = 1$. Differentiating implicitly once, we have

$$2\sin x \cos x - 2\cos y \sin y \frac{dy}{dx} = 0.$$

Differentiating implicitly again, we have

$$2\cos^2 x - 2\sin^2 x + 2\sin^2 y\left(\frac{dy}{dx}\right)^2 - 2\cos^2 y\left(\frac{dy}{dx}\right)^2 - 2\cos y \sin y \frac{d^2y}{dx^2} = 0.$$

**Example:** We are to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $(5,2)$ for the function given by the relation $x^2 - 4y^2 = 9$. Differentiating once gives

$$2x - 8y\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = \frac{x}{4y}.$$

Differentiating again gives

$$\frac{d^2y}{dx^2} = \frac{4y - 4x\frac{dy}{dx}}{16y^2}.$$

At the point in question, $\frac{dy}{dx} = \frac{5}{8}$ and

$$\frac{d^2y}{dx^2} = \frac{8 - 20 \cdot \frac{5}{8}}{64} = -\frac{9}{128}.$$

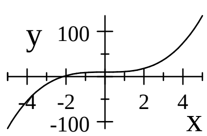
## 10. The second derivative and concavity

**1)** There are two ways to say that the graph of a function $f$ is **concave up** on an interval $I$. One is that the secant line between any two points of the graph lies above the graph and the tangent line at any point lies below the graph. The other is to speak in terms of the tangent lines at points $(x, f(x))$: If the second derivative exists and is strictly positive, that is, the slope of tangent lines is increasing at all points of $I$, the graph of the function is concave up on $I$.

**2)** There are two ways to say that the graph of a function $f$ is **concave down** on $I$. One is that the secant line between any two points of the graph lies below the graph and the tangent line at any point lies above the graph. The other is to speak of the tangent lines at points $(x, f(x))$: If the second derivative exists and is strictly negative, that is, the slope of tangent lines is decreasing at all points of $I$, the graph of the function is concave down on $I$.

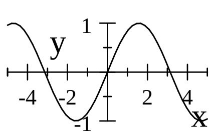
**3)** If the concavity changes at a point $c$ in $I$, for example, if $f''(x)$ changes from positive to negative or from negative to positive at some point $c$ in $I$, then $c$ is called an **inflection point** of $f$ or of the graph of $f$. At an inflection point, the graph changes from concave up to concave down or from concave down to concave up. At an inflection point, either the second derivative is 0 or it does not exist.

EXAMPLE: $f(x) = x^3+1$ has a graph that is concave down for $x < 0$ and concave up for $x > 0$.




$x^3 + 10$

Example: If $f(x) = \sin x$, the second derivative is $f''(x) = -\sin x$, so where sin is positive it is concave down and where sin is negative, it is concave up. Every point where $\sin x = 0$ is an inflection point.




$\sin x$

## 11. Methods to classify critical points

Given a critical point, we need ways to determine if they are local maxima or local minima or neither.

FIRST DERIVATIVE TEST at endpoints:

1) At a left endpoint $a$, if for some $\delta > 0$, $f'(x) > 0$ for $a < x < a + \delta$, then $a$ is a local minimum.

2) At a left endpoint $a$, if for some $\delta > 0$, $f'(x) < 0$ for $a < x < a + \delta$, then $a$ is a local maximum.

3) At a right endpoint $b$, if for some $\delta > 0$, $f'(x) > 0$ for $b - \delta < x < b$, then $b$ is a local maximum.

4) At a right endpoint $b$, if for some $\delta > 0$, $f'(x) < 0$ for $b - \delta < x < b$, then $b$ is a local minimum.

FIRST and SECOND DERIVATIVE TESTS at inside points: Suppose $c$ is a point in the domain of $f$ such that $f$ is defined on an interval $(c - \gamma, c + \gamma)$ for some $\gamma > 0$, and $f'$ is also defined on $(c - \gamma, c + \gamma)$ except perhaps at $c$. Also suppose $c$ is a critical point for $f$; that is, either $f'(c)$ does not exist or $f'(c) = 0$.

1) The function $f$ has a local maximum value at $c$ if either of the following two conditions hold:

a) For some $\delta > 0$, $f'(x) > 0$ for all $x$ in $(c - \delta, c)$ and $f'(x) < 0$ for all $x$ in $(c, c + \delta)$, or

b) $f'(c) = 0$ and $f''(c) < 0$.

2) The function $f$ has a local minimum value at $c$ if either of the following two conditions hold:

a) For some $\delta > 0$, $f'(x) < 0$ for all $x$ in $(c - \delta, c)$ and $f'(x) > 0$ for all $x$ in $(c, c + \delta)$, or

b) $f'(c) = 0$ and $f''(c) > 0$.

3) The function $f$ has **neither** a local maximum nor a local minimum at $c$ if $f'$ is strictly positive on both sides of $c$ or if $f'$ is strictly negative on both sides of $c$.

4) If $f'(c) = 0$ and $f''(c) = 0$, then the second derivative test gives no information.

**Proof.** The proof for the first derivative tests follows from the fact that a function is strictly increasing on an interval where the derivative is positive, and it is strictly decreasing on an interval where the derivative is negative.

For the second derivative test, we use the fact that if $f'(c) = 0$, $f'(c + \Delta x) = f''(c) \cdot \Delta x + E(\Delta x) \cdot \Delta x$ where $E$ is a function of $\Delta x$ with limit 0 at 0. For small values of $\Delta x$, $f'(c + \Delta x)$ is approximately equal to $f''(c) \cdot \Delta x$. If $f''(c) < 0$, then for small negative values of $\Delta x$, $f'(c + \Delta x) < 0$, and for small positive values of $\Delta x$, $f'(c + \Delta x) > 0$. That is, the derivative is positive to the left of $c$ and negative to the right of $c$. By the first derivative test, $f$ has a local maximum value at $c$. Similarly, if $f'(c) = 0$ and $f''(c) > 0$, $f$ has a local minimum value at $c$.

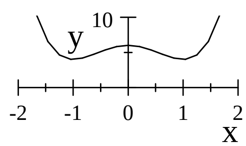
**EXAMPLE:** Let $f(x) = x^2$. Then $f'(0) = 0$ and $f''(0) = 2$, so 0 is a local minimum value of $f$. It is also the absolute minimum value.

**EXAMPLE:** Let $f(x) = x^3$. Then $f'(0) = 0$ and $f''(0) = 0$, but the function has neither a local maximum nor a local minimum value at 0.

**EXAMPLE:** Let $f(x) = x^4$. Then $f'(0) = 0$ and $f''(0) = 0$, and the function has a local minimum value at 0.

**Example:** We are to find the critical points and local maxima and minima for $f(x) = 2x^4 - 4x^2 + 6$ on the real line.

There are no end points, and no points where the derivative does not exist. The derivative $f'(x) = 8x^3 - 8x = 0$ when $x = 0$ and when $x^2 = 1$, i.e. $x = \pm 1$. To test these three points, we note that $f'(-2) < 0$, $f'(-1/2) > 0$, $f'(1/2) < 0$, and $f'(2) > 0$, so $f$ has a local minimum at $-1$ with $f(-1) = 4$, $f$ has a local maximum at 0 with $f(0) = 6$, and $f$ has a local minimum at 1 with $f(1) = 4$. We can also check this using the second derivative $f''(x) = 24x^2 - 8$. Since $f''(-1) > 0$, and $f''(1) > 0$, $-1$ and 1 are local minima of $f$. Since $f''(0) < 0$, 0 is a local maximum of $f$. Notice that $f(x)$ goes to $+\infty$ as $x$ goes to either $+\infty$ or $-\infty$. It follows that there is no absolute maxima value for the function, but the value 4 achieved at $-1$ and $+1$ is an absolute minimum value.




$2x^4 - 4x^2 + 6$

**Example:** We are to find the critical points and local maxima and minima for $f(x) = \left(\frac{x-2}{x+2}\right)^3$ on the real line minus the point $x = -2$ where the function is not defined. Since $-2$ is not a point in the domain of the function, it is not a critical point. There are no endpoints and no points where the derivative does not exist in the domain. The derivative

$$f'(x) = 3\left(\frac{x-2}{x+2}\right)^2 \cdot \frac{(x+2)-(x-2)}{(x+2)^2} = 12\frac{(x-2)^2}{(x+2)^4}$$

is 0 when $x = 2$, so this is the only critical point in the domain. To classify this point, it is easiest to use the first derivative test. The first derivative is always nonnegative, so $x = 2$ is neither a local minimum nor a local maximum.

**Example:** We want to prove that an $n$-th degree polynomial defined on the real line has at most $n - 1$ local extreme points. What we show is that an $n$-th degree polynomial has at most $n - 1$ critical points. Since a polynomial has all derivatives, this means that the derivative has at most $n - 1$ roots. But the derivative has degree $n - 1$, and so this is true.