1. Problems with entering homework

If you believe that you have entered a correct answer for the homework that the website will not accept, be sure you have carefully read the problem and the format needed. If you have done this, you should print out the problem with your solution from the screen and hand it in to your TA. If he agrees with you, he will note the need for the addition of a point to the score for that homework, and add that point at the end of the term.

I have asked the publisher to fix the problems with the website that we did not have last year.

2. A Tutoring Room is Open

7–9 p.m, Monday, Tuesday, Wednesday, Thursday, Room 140 Lincoln Hall.

3. Homework 14 due Thursday, October 12 at 9 a.m.

Section 4.2: #6, 10, 14, 26, 32, 40, 42, 48.

Section 4.3: #2, 4.

4. Homework 15 due Tuesday, October 17 at 9 a.m.

Section 4.3: #6, 16, 20, 26, 36.

Section 4.4: #6, 10, 14, 32, 38.

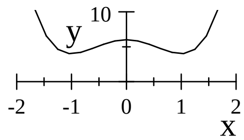
5. Written problem for next week

Work Problem 46 in Section 4.3 of your text (Page 246).

6. Examples for local maxima and minima

Sample Question: Find the critical points and local maxima and minima for $f(x) = 2x^4 - 4x^2 + 6$ on the real line.

There are no end points, and no points where the derivative does not exist. The derivative $f'(x) = 8x^3 - 8x = 0$ when $x = 0$ and when $x^2 = 1$, i.e. $x = \pm 1$. To test these three points, we note that $f'(-2) < 0$, $f'(-1/2) > 0$, $f'(1/2) < 0$, and $f'(2) > 0$, so f has a local minimum at -1 with $f(-1) = 4$, f has a local maximum at 0 with $f(0) = 6$, and f has a local minimum at 1 with $f(1) = 4$. We can also check this using the second derivative $f''(x) = 24x^2 - 8$. Since $f''(-1) > 0$, and $f''(1) > 0$, -1 and 1 are local minima of f. Since $f''(0) < 0$, 0 is a local maximum of f. Notice that $f(x)$ goes to $+\infty$ as x goes to either $+\infty$ or $-\infty$. It follows that there is no absolute maxima value for the function, but the value 4 achieved at -1 and $+1$ is an absolute minimum value.



$2x^4 - 4x^2 + 6$

Example: An n-th degree polynomial defined on the real line has at most $n-1$ local extreme points since all derivatives exist and the derivative is of degree $n-1$. That is, the derivative can have at most $n-1$ roots.

7. Finding where a function is positive and where it is negative

We want to plot **continuous** functions defined on the real line. For polynomials, at least, the behavior at $+\infty$, and $-\infty$ is determined by the term of highest degree. The next thing to determine is where the function is positive and where it is negative.

FIRST: Find all zeros of the function.

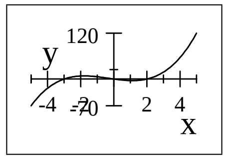
SECOND: If there are only a finite number of zeros, evaluate the function at some point below the first zero (or note the behavior at $-\infty$), and at some point above the last zero (or note the behavior at $+\infty$). Also evaluate the function at some point in each interval bounded by zeros. By the intermediate value theorem, the function cannot change sign between zeros, so now you know the sign of the function at every point of the real line.

THIRD: Sketch the real line showing the points where the function is 0 and indicating with pluses and minuses the sign on the intervals determined by the zeros. This lets you know where the plot of the function should cross the x-axis, where it should be above the x-axis, and where it should be below the x-axis.

NOTE: If you have a factored polynomial, then by evaluating the product at one point in the interval between zeros, you can determine the sign of each factor in that interval and thus the sign of the product.

EXAMPLE: $f(x) = x^3 + x^2 - 6x = x(x-2)(x+3)$. The zeros are at -3, 0, and 2. Since $\lim_{x\to-\infty} f(x) = -\infty$, we know the function is negative below the point $x = -3$. Also, $f(-4) = -64 + 16 + 24 = -24$ gives the same information. Since $f(-1) = -1 + 1 + 6 = 6$, we know the function is positive in the interval $(-3, 0)$. We can also note that at $x = -1$, two of the factors are negative and one factor is positive, so the function is positive at -1. Since $f(1) = 1 + 1 - 6 = -4$, we know the function is negative in the interval $(0, 2)$. Since $\lim_{x\to+\infty} f(x) = +\infty$, we know the function is positive above the point $x = 2$. We also know this because $f(3) = 27 + 9 - 18 = 36$. Rather than evaluating at 3, we can also note that each of the factors is positive there, so the function is positive there.

$x^3 + x^2 - 6x$



8. FINDING WHERE THE FUNCTION IS INCREASING AND WHERE IT IS DECREASING.

Assume now that the derivative is defined at every point. This will be the case for a polynomial. The next thing to do is find points where the derivative is 0 and determine the sign of the derivative on the intervals determined by those zeros. We know the function is increasing where the derivative is positive and decreasing where the derivative is negative. We also know the tangent line is horizontal at points where the derivative is 0. Just as we did for the function itself, we now sketch the real line showing the zeros of the derivative and the intervals where the function is increasing and the intervals where it is decreasing. These two sketches, one for the function and one for the derivative, help us graph the function.

Example: For $f(x) = x^2$, $f'(x) = 2x$. Of course, $f'(0) = 0$. Since f' is negative for points below 0, the function is decreasing below 0. Since f' is positive for points above 0, the function is increasing above 0.

Example: For $f(x) = x^3 + x^2 - 6x$,

$$f'(x) = 3x^2 + 2x - 6 = 0$$

when $x = -1.78$ and when $x = 1.12$. Since $\lim_{x\to-\infty} f'(x) = +\infty$, f' is positive, i.e., f is increasing, at points below -1.78. Since $f'(0) = -6$, f' is negative, i.e., f is decreasing on the interval $(-1.78, 1.12)$. Since $\lim_{x\to+\infty} f'(x) = +\infty$, f' is positive, i.e., f is increasing, at points above 1.12.

9. FINDING WHERE THE FUNCTION IS CONCAVE UP AND WHERE IT IS CONCAVE DOWN

Assume now that the second derivative is defined at every point. This will be the case for a polynomial. The next thing to do is find points where the second derivative is 0 and determine the sign of the second derivative on the intervals determined by those zeros. We know the graph of the function is concave up where the second derivative is positive and concave down where the second derivative is negative. Just as we did for the function and derivative, we now sketch the real line showing the zeros of the second derivative and the intervals where the graph is concave up and the intervals where it is concave down. These three sketches, one for the function, one for the derivative and one for the second derivative, help us graph the function.

Example: For $f(x) = x^3 + x^2 - 6x$,

$$\begin{aligned} f'(x) &= 3x^2 + 2x - 6 \\ f''(x) &= 6x + 2 = 0 \text{ when } x = -1/3. \end{aligned}$$

Since f'' is negative below this point and positive above it, $(-1/3, f(-1/3))$ is an inflection point on the graph.

10. LIMITS AT INFINITY

We say a function f defined on an interval $(a, +\infty)$ has a finite limit L at $+\infty$, and we write $\lim_{x\to+\infty} f(x) = L$ if $f(x) = L + E(x)$ where $E(1/x)$ has limit 0 as x goes to 0 through positive values. We say a function f defined on an interval $(-\infty, a)$ has a finite limit L at $-\infty$, and we write $\lim_{x\to-\infty} f(x) = L$ if $f(x) = L + E(x)$ where $E(1/x)$ has limit 0 as x goes to 0 through negative values.

Example: $2 + 1/x^3$ has limit 2 at both infinities. The error $E(x) = 1/x^3$ has the property that $E(1/x) = 1/(1/x)^3 = x^3$, which has limit 0 from both sides of 0.

11. BEHAVIOR OF POLYNOMIALS AT $+\infty$ AND $-\infty$

We will say that x goes to $+\infty$, if we look at increasingly larger values of x; and x goes to $-\infty$ if x is negative while $|x|$ goes to $+\infty$. To see what happens to a polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

as x goes to $+\infty$ and $-\infty$, we assume $n \geq 1$ and $a_n \neq 0$, we factor out x^n, and we write the polynomial as follows

$$f(x) = x^n \left(a_n + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \right).$$

As x goes to $+\infty$ and $-\infty$, all of the terms go to 0 except the constant term a_n, and so the polynomial looks more and more like $a_n x^n$. This means that if n is odd, and $a_n > 0$, then $f(x)$ becomes larger and larger as x goes to $+\infty$, and $f(x)$ is negative while $|f(x)|$ becomes larger and larger as x goes to $-\infty$. we write

$$\lim_{x\to-\infty} f(x) = -\infty, \quad \text{and} \quad \lim_{x\to+\infty} f(x) = +\infty. \tag{1}$$

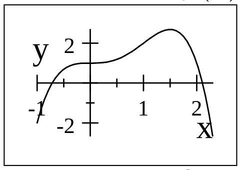
On the other hand, if n is odd and $a_n < 0$, then we have

$$\lim_{x\to-\infty} f(x) = +\infty, \quad \text{and} \quad \lim_{x\to+\infty} f(x) = -\infty. \tag{2}$$

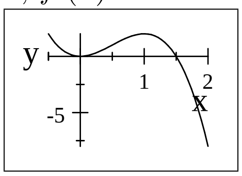
Example:

$$f(x) = 3x^5 - 2x^2 + 4x - 7 = x^5 \left(3 - \frac{2}{x^3} + \frac{4}{x^4} - \frac{7}{x^5} \right),$$

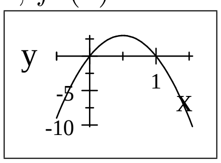
Example: $f(x) = 1 + 2x^3 - x^4$.



Here, $f'(x) = 6x^2 - 4x^3$.



Also, $f''(x) = 12x - 12x^2$.



The function $f(x)$ is positive approximately between $-.7$ and 2.2. Also $f'(x) = 0$ when $x = 0$ and $x = \frac{3}{2}$. Moreover, f' is positive below 0 and between 0 and $\frac{3}{2}$, and it is negative above $\frac{3}{2}$. Thus the function is increasing below $\frac{3}{2}$ and decreasing after that. f'' is negative below 0 and above 1, so the function is concave down on those intervals. f'' is positive between 0 and 1, so the function is concave up on that interval. There is an inflection point at 0 and 1.

12. Behavior of ratios of polynomials at infinity

To find the behavior of a ratio of polynomial

$$\frac{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0}{b_m x^m + b_{m-1} x^{m-1} + \ldots + b_0}$$

at infinity, divide numerator and denominator by $b_m x^m$ so that the denominator tends to 1.

1) If $m > n$, the numerator will then tend to 0, and so the x-axis $y = 0$ is a "horizontal" asymptote of the graph of the function. As an example, consider

$$\frac{x}{2x^2 + 2x + 1} = \frac{\frac{1}{2x}}{1 + \frac{1}{x} + \frac{1}{2x^2}} \to 0.$$

at both $+\infty$ and at $-\infty$.

2) If $m = n$, then the new numerator will equal a_n/b_m plus some terms which go to 0 as x goes to $+\infty$ and $-\infty$. We then say that the line $y = a_n/b_m$ is a horizontal asymptote of the graph of the function. As an example, consider

$$\frac{3x^2+2}{2x^2+2x+1} = \frac{\frac{3}{2}+\frac{1}{x^2}}{1+\frac{1}{x}+\frac{1}{2x^2}} \to \frac{3}{2}.$$

at both $+\infty$ and at $-\infty$.

3) If after division the numerator equals $(a_n/b_m)x + a_{n-1}/b_m$ plus some terms which go to zero at $+\infty$ and $-\infty$, then we say that the line $y = (a_n/b_m)x + a_{n-1}/b_m$ is an asymptote of the graph of the function. As an example, consider

$$\frac{3x^3+4x^2+5}{2x^2+2x+1} = \frac{\frac{3}{2}x+2+\frac{5}{2x^2}}{1+\frac{1}{x}+\frac{1}{2x^2}} \to \frac{3}{2}x+2.$$

at both $+\infty$ and at $-\infty$. The line $\frac{3}{2}x+2$ is a slant asymptote of the graph.

4) Otherwise, the ratio still tends to $+\infty$ or $-\infty$ but has no linear asymptotes.

13. Horizontal and Vertical asymptotes

If a function tends to a value b as x goes to $+\infty$ or $-\infty$, then we say that the horizontal line $y = b$ is a horizontal asymptote of the graph of the function or briefly, of the function.

EXAMPLES: $f(x) = 1/(1+x^2)$ has the y-axis, $y = 0$, as a horizontal asymptote. The function arctan has $-\pi/2$ and $\pi/2$ as horizontal asymptotes.

If the function tends to $+\infty$, or $-\infty$, at a point a, then we say it has a vertical asymptote at a. This is also what we say if the function tends to $+\infty$ or $-\infty$ from just one side, or to $+\infty$ from one side and $-\infty$ from the other.

EXAMPLE: $f(x) = 1/(x-1)$ has a vertical asymptote at $x = 1$.