1. COME TO CLASS

Starting next time, we will cover material in a way that is not in the book. Come to class.

2. A TUTORING ROOM IS OPEN

7–9 p.m, Monday, Tuesday, Wednesday, Thursday, Room 140 Lincoln Hall.

3. HOMEWORK 15 DUE TUESDAY, OCTOBER 17 AT 9 A.M.

Section 4.3: #6, 16, 20, 26, 36.

Section 4.4: #6, 10, 14, 32, 38.

4. HOMEWORK 16 DUE THURSDAY, OCTOBER 19 AT 9 A.M.

Section 4.5: #8, 10, 18, 28.

Section 4.6: #24, 30, 32, 78, 80, 82

5. WRITTEN PROBLEM FOR NEXT WEEK

Work Problem 46 in Section 4.3 of your text (Page 246).

6. INDETERMINANT FORMS AND L'HÔPITAL'S RULE

At times, one needs to determine a limit involving two functions where the ratio of the limits is $0/0$, or ∞/∞, or the difference is $\infty - \infty$. Here, the usual rules do not apply. We used the limit of the indeterminant form $\sin x/x \to 1$ at $x = 0$ to figure out how to differentiate trigonometric functions. For that limit, we used geometric arguments. In general, we can use l'Hôpital's Rule if we already know the derivatives. Here is the rule for the limit of a quotient where the numerator and denominator both have limits equal to 0.

Theorem 1. *Suppose f and g are differentiable (therefore continuous) in some open interval containing a point a, and g and and its derivative g' are never equal to 0 in that interval except perhaps at a itself. If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the latter limit exists as a finite number or as $+\infty$ or $-\infty$.

EXAMPLE:

$$\lim_{x\to 0} \frac{\sin x}{\tan x} = \lim_{x\to 0} \frac{\cos x}{\sec^2 x} = \lim_{x\to 0} \cos^3 x = 1.$$

Proof. If $g'(a) \neq 0$, then since both f and g are continuous at a, they both equal 0 at a, so for Δx small but not 0, there are functions E and F with limit 0 at 0 such that

$$\begin{aligned}\frac{f(a+\Delta x)}{g(a+\Delta x)} &= \frac{f(a)+f'(a)\Delta x+E(\Delta x)\Delta x}{g(a)+g'(a)\Delta x+F(\Delta x)\Delta x}\\ &= \frac{f'(a)+E(\Delta x)}{g'(a)+F(\Delta x)} \to \frac{f'(a)}{g'(a)}.\end{aligned}$$

If $g'(a)$ is 0 but $f'(a) \neq 0$, we get a positive or negative infinite limit.

If all derivatives of both f and g are 0 up to the $n-1^{\text{st}}$ derivative and the next two derivatives exists, then you will see in the next course (Page 709) that for Δx small but not 0, there are functions E and F with limit 0 at 0 such that

$$\begin{aligned}\frac{f(a+\Delta x)}{g(a+\Delta x)} &= \frac{\frac{f^{(n)}(a)}{n!}(\Delta x)^n + E(\Delta x)(\Delta x)^n}{\frac{g^{(n)}(a)}{n!}(\Delta x)^n + F(\Delta x)(\Delta x)^n}\\ &= \frac{f^{(n)}(a)+n!E(\Delta x)}{g^{(n)}(a)+n!F(\Delta x)}.\end{aligned}$$

If $g^{(n)}(a) \neq 0$, the limit is $f^{(n)}(a)/g^{(n)}(a)$. If $g^{(n)}(a) = 0$ but $f^{(n)}(a) \neq 0$ the limit is $+\infty$ or $-\infty$. This means that if $f'(a) = 0$ and $g'(a) = 0$, you keep taking derivatives of both the numerator and denominator until one or the other of the derivatives is not zero, and then that limit is also the original limit. One must be sure that the denominator is not 0 near a except at a.

Example:

$$\lim_{x\to 0}\frac{2x^2}{x\sin x} = \lim_{x\to 0}\frac{4x}{\sin x + x\cos x} = \lim_{x\to 0}\frac{4}{\cos x+\cos x - x\sin x} = 2.$$

L'Hôpital's Rule will also work if the limits of f and g at a are both infinite. It also works for one sided limits.

Example:

$$\lim_{x\to 0+}\frac{\ln 3x}{\ln x} = \lim_{x\to 0+}\frac{\frac{3}{3x}}{\frac{1}{x}} = 1.$$

L'Hôpital's Rule will also work at $+\infty$ and $-\infty$ since these limits can be made into limits at 0 by replacing the independent variable x with $1/t$ and letting t go to 0.

Example:

$$\lim_{x\to+\infty}\frac{x}{e^x} = \lim_{x\to+\infty}\frac{1}{e^x} = \lim_{x\to+\infty}e^{-x} = 0.$$

In fact, for any positive integer n, $\lim_{x\to+\infty} \frac{x^n}{e^x} = 0$. This means that e^x goes to ∞ faster than any polynomial. on the other hand,

$$\lim_{x\to+\infty} \frac{\ln x}{x^n} = \lim_{x\to+\infty} \frac{1}{nx^n} = 0.$$

This means that $\ln x$ goes to ∞ more slowly than any power of x.

An indeterminant form with limit $0 \cdot \infty$ can be put into the form $0/0$ by dividing.

Example:

$$\lim_{x\to 0+} xe^{1/x} = \lim_{x\to 0+} \frac{e^{1/x}}{1/x} = \lim_{x\to 0+} \frac{\left(D_x \frac{1}{x}\right) e^{1/x}}{\left(D_x \frac{1}{x}\right)} = \lim_{x\to 0+} e^{1/x} = +\infty.$$

Other indeterminant forms can be resolved by putting a difference over a common denominator or taking ln and then working with that.

Example:

$$\begin{aligned} \lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{\sin x}\right) &= \lim_{x\to 0} \frac{\sin x - x}{x \sin x} = \lim_{x\to 0} \frac{\cos x - 1}{\sin x + x\cos x} \\ &= \lim_{x\to 0} \frac{-\sin x}{2\cos x - x \sin x} = 0. \end{aligned}$$

Some problems can be handled in two ways, one using the rule for the ratio of polynomials, the other using l'Hôpital's Rule.

For example,

$$\lim_{x\to+\infty} \frac{3x^2 - 2x + 4}{5x^2 - 4x} = \lim_{x\to+\infty} \frac{\frac{3}{5} - \frac{2}{5}\frac{1}{x} + \frac{4}{5}\frac{1}{x^2}}{1 - \frac{4}{5}\frac{1}{x}} = \frac{3}{5}.$$

Using l'Hôpital's Rule, we have

$$\lim_{x\to+\infty} \frac{3x^2 - 2x + 4}{5x^2 - 4x} = \lim_{x\to+\infty} \frac{6x - 2}{10x - 4} = \lim_{x\to+\infty} \frac{6}{10} = \frac{3}{5}.$$

It is in Section 4.8 that you start using l'Hôpital's Rule.

7. ANTIDIFFERENTIATION

For many "real-life" problems, one knows the rate of change of a function and the value of the function at a point. The problem then, is to find the function itself. Such a function is called an antiderivative of the original function. We have seen that all antiderivatives of a given function on an interval differ by just a constant. For example, all functions with derivative $2x$ on the real line have the form x^2 plus a constant.

The class of function whose derivative is a given function f is called **the** antiderivative or indefinite integral of f. Each function whose derivative is f is called **an** antiderivative of f. The problem of finding the antiderivative is usually posed by giving the differential and using the "integral sign". For example

$$\int x^2dx = \frac{1}{3}x^3 + C.$$

If you are just finding the antiderivative, you must put in the constant C in your answer, since more than one function has derivative x^2. Of course, if you also know what you want the value of your antiderivative to be at a point, you can solve for C.

If you are working on two different intervals, then you can change the value of C when you go from one interval to the other.

EXAMPLE: For $x \neq 0$,

$$\int \frac{-1}{x^2}dx = \frac{1}{x} + C_1 \text{ for } x > 0 \text{ and } = \frac{1}{x} + C_2 \text{ for } x < 0.$$

8. RULES FOR FINDING ANTIDERIVATIVES

Here are some rules:

$$\int x^r dx = \frac{x^{r+1}}{r+1} + C \quad \text{for } r \neq -1.$$

Note that $\int 1\, dx = \int x^0\, dx = x + C$.

Here are some other rules:

$$\begin{aligned}
\int \sin x\, dx &= -\cos x + C, \quad \int \cos x\, dx = \sin x + C\\
\int \sec^2 x\, dx &= \tan x + C, \quad \int \csc^2 x\, dx = -\cot x + C\\
\int \sec x \tan x\, dx &= \sec x + C, \quad \int \csc x \cot x dx = -\csc x + C\\
\int e^x dx &= e^x + C, \qquad \int \frac{1}{x}dx = \ln|x| + C.
\end{aligned}$$

In the last case, the constant can change when you cross from positive x to negative x.

We also have the Sum Rule:

$$\int f(x) + g(x)\, dx = \int f(x)\, dx + \int g(x)\, dx,$$

and the rule for multiplication by a real number:

$$\int a \cdot f(x)\,dx = a \int f(x)\,dx \qquad \text{for each real number } a.$$

The sum rule says that you can simplify your work by breaking up a sum. For example

$$\begin{aligned} \int x^4 + x^3 + 1\,dx &= \int x^4\,dx + \int x^3\,dx + \int dx \\ &= \frac{1}{5}x^5 + \frac{1}{4}x^4 + x + C. \end{aligned}$$

Notice that here we have combined the constants from each of the three integrals into one constant.

The rule for multiplying by a constant means that you can simplify your work by moving a constant outside of the integral. For example:

$$\int \pi x\,dx = \pi \int x\,dx = \pi(\frac{1}{2}x^2 + C) = \frac{\pi}{2}x^2 + C.$$

Notice for the indefinite integral, we have “absorbed” π into the constant C. That is, since C is an arbitrary constant, we do not have to write πC; we just rename πC and call it C.