1. Come to Class

Starting next time, we will cover material in a way that is not in the book. Come to class.

2. A Tutoring Room is Open

7–9 p.m, Monday, Tuesday, Wednesday, Thursday, Room 140 Lincoln Hall.

3. Homework 15 due Tuesday, October 17 at 9 a.m.

Section 4.3: #6, 16, 20, 26, 36.

Section 4.4: #6, 10, 14, 32, 38.

4. Homework 16 due Thursday, October 19 at 9 a.m.

Section 4.5: #8, 10, 18, 28.

Section 4.6: #24, 30, 32, 78, 80, 82

5. Written problem for this week

Work Problem 46 in Section 4.3 of your text (Page 246).

6. More on AntiDifferentiation

One reason we use differential notation in asking for antiderivatives is because the notation imbeds the chain rule. Here is an example:

$$\int (x^2+5x+2)^3 \cdot (2x+5)\,dx = \frac{1}{4}(x^2+5x+2)^4 + C.$$

Once I give you the answer, you can check that it is right. [Note, you should always check your answers by differentiation.] However, here is a way to "feel" your way to a solution. If we set $u = x^2+5x+2$, then the differential $du = (2x+5)\,dx$. With this change of variables, we want to find $\int u^3\,du$. The solution is

$$\frac{1}{4}u^4 + C = \frac{1}{4}(x^2+5x+2)^4 + C.$$

When you change variables in this way for finding antiderivatives, a.k.a., **indefinite integrals**, you must change back to the original variable for the final answer.

EXAMPLE: Find $\int \sin 3x\,\cos 3x\,dx$. If we set $u = \sin 3x$, then the differential $du = 3\cos 3x\,dx$, so $\frac{1}{3}du = \cos 3x\,dx$. Thus we have

$$\int \sin 3x\,\cos 3x\,dx = \frac{1}{3}\int u\,du = \frac{1}{6}u^2 + C = \frac{1}{6}\sin^2 3x + C.$$

Sample Problem Find $\int \cos(x^2+4x+1)\,(x+2)\,dx$. Here, we let $u = x^2+4x+1$, so that $du = (2x+4)\,dx$, and $\frac{1}{2}du = (x+2)\,dx$. In terms of u, we want $\frac{1}{2}\int \cos u\;du =$

$\frac{1}{2}\sin u + C$. The answer, which must be in terms of x, is then $\frac{1}{2}\sin(x^2+4x+1)+C$. You might at the outset guess that the answer takes the form $\sin(x^2+4x+1)+C$. If you check this guess, you see that you must multiply it by $\frac{1}{2}$, and then you have the right answer. Of course, you "absorb" the factor $\frac{1}{2}$ into the constant C, and write C instead of $\frac{1}{2}C$, since C itself is arbitrary.

EXAMPLE:

$$\int e^{e^{e^{e^x}}} e^{e^{e^x}} e^{e^x} e^x dx = e^{e^{e^{e^x}}} + C.$$

NOTE: For antedifferentiating, you will usually want to check your answer by differentiating it to see if it is right.

7. Velocity and Acceleration

For these problems, one must usually find the position $x(t)$ or $y(t)$ as a function of time when one is given the acceleration $a(t)$ as a function of time. If you think of the motion as horizontal, you should think of a position $x(t)$ along the x-axis. If you think of the motion as vertical, you should take the position as $y(t)$ along the y-axis.

The sign convention is important. velocity is speed with a sign convention. For example, if the value of $x(t)$ is increasing, then the velocity $v(t) = x'(t) \geq 0$, but if the value of $x(t)$ is decreasing, then the velocity $v(t) = x'(t) \leq 0$. If the velocity $v(t)$ is increasing, then the acceleration $a(t) = v'(t) \geq 0$, but if the velocity $v(t)$ is decreasing then the acceleration $a(t) = v'(t) \leq 0$.

EXAMPLE: Assume a particle is traveling on the x-axis starting at time $t = 0$ at a point x_0 with an initial velocity $v_0 > 0$. This means that whatever the value of x_0, positive or negative or 0, the value of $x(t)$ is increasing at time $t = 0$. Now also assume that the particle experiences a constant negative acceleration a. This means the particle is experiencing a force that will decrease its positive velocity to 0 and then, if the force continues to work, will send the particle faster and faster towards $-\infty$. Such a force could be caused, for example, by a constant braking of the particle (or automobile); the process will stop then when the velocity is 0. We now have all the information we need to find the position $x(t)$ at any time t. First, $v(t)$ is that antiderivative of the constant function $a(t) = a$ such that $v(0) = v_0$. That is,

$$v(t) = a \cdot t + v_0.$$

Second, the position $x(t)$ is that antiderivative of $v(t)$ such that $x(0) = x_0$. That is,

$$x(t) = \frac{a}{2}t^2 + v_0 t + x_0.$$

Notice that for times close to 0, $v_0 t > \frac{a}{2}t^2$. However, as time goes on, the negative term $\frac{a}{2}t^2$ will "overpower" the term $v_0 t$. (Of course, if $a > 0$, then the particle will go faster and faster to $+\infty$.) If we want to know how big x gets before the particle turns around, we solve the value of t that makes $v(t) = 0$, i.e., $-v_0/a$ (remember, $a < 0$.)

A similar story is true for vertical motion $y(t)$ with an initial velocity v_0 and initial position x_0, whatever their signs might be. If the acceleration is a constant function $a(t) = a$, then at any later time (before the experiment is stopped),

$$y(t) = \frac{a}{2}t^2 + v_0 t + x_0.$$

In particular, for a particle falling near the Earth and assuming that the only force is the force due to gravity, we take the positive y-axis as "up" and set $a(t) = -32$ feet/sec^2 or -980 cm/sec^2. Notice that these calculations have been made under the assumption of a constant acceleration.

For acceleration that is not constant, you must take the antiderivative twice to find the position at any time t.

Sample Problem: A ball is tossed up from the edge of a 96 foot building with an initial velocity of 16 feet per second. Find the velocity and height of the ball at any time t, the time when the ball reaches its highest point, and the time when the ball hits the ground.

Answer: The acceleration at any time t is $a(t) = -32$ feet/sec^2, the velocity $v(t) = -32t + 16$ feet/sec, and the height $y(t) = -16t^2 + 16t + 96$. The maximum value of y is when $v(t) = 0$, that is at $1/2$ of a second. The ball hits when ground when $t^2 - t - 6 = 0$, that is, $t = -2$ and $t = 3$ seconds; only the second answer is correct since it must be positive.

8. Differential Equations

Differential equations involve not only the independent variable, but also the function and various derivatives. The simplest ones are of the form $f'(x) = g(x)$, and then the general solution is $f(x) = \int g(x)dx + C$. If we also specify a value for the solution at a particular point, then we can solve for C and get a single solution.

9. Approximating by sums

Suppose you are given an onion and you a very sharp knife and a very good ruler. You want to estimate the volume of the onion. You can cut the onion up into thin slices, use the radius of either face of a slice to estimate the area, multiply by the thickness of the slice, and add up these estimates to estimate the volume of the onion. If the cutting process losses no onion, you hope that when the slices are thin enough, your estimate for the volume of the onion is close to the actual value. That is, you hope that the sum of the errors you make with each estimation goes to zero as the thickness of the slices goes to 0.

Another example is finding the total force of water on a dam when the distance across changes with the height. We can cut up the dam into thin horizontal strips and add up the forces on the strips. It is this kind of cutting up, estimating, and adding up that we will use for the rest of the course. We need some notation for the adding up.

10. Summation Notation

For the last half of this course, we are going to talk about sums and limits of sums. For this we need some new notation. Suppose we want to write the sum of the first 30 positive integers. Of course, we can write

$$1 + 2 + 3 + \cdots + 28 + 29 + 30,$$

but this is too long and it is cumbersome. We use instead, "summation notation" involving the Greek letter capitol sigma, $\sum$, which is the equivalent of the letter S. We let i be a variable and we write

$$\sum_{i=1}^{30} i = 1 + 2 + 3 + \cdots + 28 + 29 + 30.$$

The subscript means that the variable i takes all integer values between 1 and 30. Of course, we could just as easily write $\sum_{n=1}^{30} n$, or $\sum_{j=1}^{30} j$, or $\sum_{m=1}^{30} m$ for this sum. We can also use a variables for the first and/or last integer in the sum. For example, we can write

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + (n-1) + n. \tag{1}$$

It turns out that we can get a compact formula for the value of this sum. First, we note that

$$\sum_{i=1}^{n} i = n + (n-1) + \cdots + 3 + 2 + 1. \tag{2}$$

If we add the left sides of Equations 1 and 2, we get $2 \cdot \sum_{i=1}^{n} i$. If we add the right sides of Equations 1 and 2, we get $n(n+1)$. That is, $2 \cdot \sum_{i=1}^{n} i = n(n+1)$, so

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}.$$

I would like you to remember that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}$.

Here are two other formulas that you need not memorize:

$$\begin{aligned} \sum_{i=1}^{n} i^2 &= \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \\ \sum_{i=1}^{n} i^3 &= \frac{n^2(n+1)^2}{4} = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} = \left(\sum_{i=1}^{n} i\right)^2. \end{aligned}$$

Here are some other examples of the summation notation:

$$\sum_{j=3}^{7} j^2 = 9 + 16 + 25 + 36 + 49.$$

For $r \neq 0$,

$$\sum_{i=0}^{5} r^i = 1 + r + r^2 + r^3 + r^4 + r^5.$$

Note that we start here with $i = 0$.

Sample Problem: Evaluate the sum $\sum_{j=1}^{3} j^4$. Ans $1^4+2^4+3^4 = 1+16+81 = 98$.

Here is a nice problem: Find a compact formula for $\sum_{i=0}^{n} r^i$ when $r \neq 1$. [Hint: write out the sum $\sum_{i=0}^{n} r^i$ and the sum $r \cdot \sum_{i=0}^{n} r^i$.]

We will need three general facts about sums. Let

$$a_1, a_2, \ldots, a_n, \text{ and } b_1, b_2, \ldots, b_n$$

be two ordered sets of n numbers. Also let c be a real number. Then

$$\sum_{i=1}^{n} a_i \; + \; \sum_{i=1}^{n} b_i = (a_1 + \cdots + a_n) + (b_1 + \cdots + b_n)$$

$$= (a_1 + b_1) + \cdots + (a_n + b_n) = \sum_{i=1}^{n} (a_i + b_i).$$

Also,

$$\sum_{i=1}^{n} c \cdot a_i = c \cdot a_1 + c \cdot a_2 + \cdots + c \cdot a_n = c \cdot (a_1 + \cdots + a_n) = c \cdot \sum_{i=1}^{n} a_i.$$

Since for any numbers a and b, $|a + b| \leq |a| + |b|$ (triangle inequality), we also have

$$\left| \sum_{i=1}^{n} a_i \right| = |a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n| = \sum_{i=1}^{n} |a_i|.$$

EXAMPLES:

$$\sum_{i=1}^{n} (3i + 4i^2) = 3 \cdot \sum_{i=1}^{n} i \; + \; 4 \cdot \sum_{i=1}^{n} i^2.$$

$$\left| \sum_{i=1}^{5} (-1)^i \right| = |-1 + 1 - 1 + 1 - 1| \leq 1 + 1 + 1 + 1 + 1 = \sum \left| (-1)^i \right|.$$