1. COME TO CLASS

We now cover material in a way that is not in the book. Come to class.

2. A TUTORING ROOM IS OPEN

7–9 p.m, Monday, Tuesday, Wednesday, Thursday, Room 140 Lincoln Hall.

3. HOMEWORK 16 DUE THURSDAY, OCTOBER 19 AT 9 A.M.

Section 4.5: #8, 10, 18, 28.

Section 4.6: #24, 30, 32, 78, 80, 82.

4. HOMEWORK 17 DUE TUESDAY, OCTOBER 24 AT 9 A.M.

Section 4.7: #2, 8, 34, 36.

Section 4.8: #8, 22, 24, 36.

Section 4.9: #6, 10.

5. WRITTEN PROBLEM FOR NEXT WEEK

Find the upper sum, the lower sum, the Riemann sum (evaluating at the left of each interval) and the value of $E_f(\Delta x)$ for the function $f(x) = x^4$ on the interval $[-1, 1]$ using $\Delta x = 3/10$. Remember, this is a sample problem for the next exam.

6. AREA UNDER GRAPHS.

Another name for the antiderivative of a function is the **indefinite integral**. We will see the reason for this later. Now we study the **definite integral** or just integral. The idea of a definite integral is most easily illustrated by the problem of finding the area bounded by the graph of a positive continuous function f, and the lines $y = 0$ (i.e., the x-axis), $x = a$, and $x = b$, when $a < b$. If $a = 1$, $b = 4$, and $f(x) = 7$ for all x, then the area is $3 \cdot 7 = 21$. If on the same interval $[1, 4]$, $f(x) = 2x - 2$, then the area is $\frac{1}{2} \cdot 3 \cdot 6 = 9$, while if $f(x) = 2x$, then the area is $\frac{1}{2}(2 + 8) \cdot 3 = 15$. The problem is, what if the graph of f is not a straight line? In this case, we can approximate the area by cutting up the interval $[a, b]$ into small intervals of length Δx and summing the product of Δx with the value of the function at the left end of each interval.

First lets look at the process of cutting up an interval $[a, b]$. **Unlike the book**, we will always cut up the interval in the same way. Given any positive number Δx, we let $x_0 = a$, $x_1 = a + \Delta x$, $x_2 = a + (2 \cdot \Delta x)$, and so on. When we come to the first n such that $a + (n \cdot \Delta x) \geq b$, we set $x_n = b$. This means that the length of all but the last of the intervals $[x_i, x_{i+1}]$ is Δx; the length of the last interval $[x_{n-1}, x_n]$ may be less than Δx.

EXAMPLE: If $a = 1$ and $b = 3$, and $\Delta x = \frac{1}{2}$, then $n = 4$, and

$$x_0 = 1,\ x_1 = 1.5,\ x_2 = 2,\ x_3 = 2.5,\ \text{and}\ \ x_4 = 3.$$

On the other hand, if $\Delta x = \frac{3}{10}$, then $n = 7$ and

$$x_0 = 1,\ x_1 = 1.3,\ x_2 = 1.6,\ x_3 = 1.9,\ x_4 = 2.2,\ x_5 = 2.5,\ x_6 = 2.8,\ \text{ and }\ x_7 = 3.$$

Here, all of the intervals have length $\frac{3}{10}$ except the last interval which has length $\frac{2}{10}$.

Now let us use this last way of cutting up the interval $[1, 3]$ to estimate the area under the graph of the function $y = x^2$ and above the interval $[1, 3]$. We will estimate by taking the area of the rectangle of height x_0^2 and base the interval $[x_0, x_1]$, and adding to that number the area of the rectangle of height x_1^2 with base equal to the interval $[x_1, x_2]$, and adding to that sum the area of the rectangle of height x_2^2 with base equal to the interval $[x_2, x_3]$, etc. In symbols, we want

$$\sum_{i=1}^{7} x_{i-1}^2 \cdot \Delta x_i$$

where $\Delta x_i = \Delta x = 3/10$, except when $i = 7$, and then, $\Delta x_7 = 2/10$. That is, we want

$$1 \cdot .3 + (1.3)^2 \cdot .3 + (1.6)^2 \cdot .3 + (1.9)^2 \cdot .3 + (2.2)^2 \cdot .3 + (2.5)^2 \cdot .3 + (2.8)^2 \cdot .2.$$

This is 7.55, which is not a very good approximation to the area, but as we make Δx, smaller and smaller, the approximation becomes increasingly better.

In general, for a continuous function f, we will see that the limit as $\Delta x \to 0$ does not depend on where we evaluate the function in each interval $[x_{i-1}, x_i]$, so unlike the book, we will always choose the left hand point x_{i-1} for our approximating sums. (There is another reason for this, which we will discuss later.) We will call this sum the **Riemann sum** for the function f on the interval $[a, b]$ with respect to Δx. We will write $R_a^b(f, \Delta x)$ for this sum. In symbols,

$$R_a^b(f, \Delta x) = \sum_{i=1}^{n} f(x_{i-1})\, \Delta x_i$$

where $\Delta x_i = \Delta x$ for $1 \leq i \leq n - 1$, and $\Delta x_n \leq \Delta x$.

7. UPPER AND LOWER SUMS

We will often use Riemann sums and two other sums associated with the partition given by Δx. Suppose a **continuous** function f and an interval $[a, b]$ are given. Also suppose that for a given Δx, m_i is the minimum value of $f(x)$ in the interval $[x_{i-1}, x_i]$ and M_i is the maximum value of $f(x)$ in the interval $[x_{i-1}, x_i]$. We know that these minima and maxima exist. We now have

$$\sum_{i=1}^{n} m_i \cdot \Delta x_i \leq \sum_{i=1}^{n} f(x_{i-1})\, \Delta x_i \leq \sum_{i=1}^{n} M_i \cdot \Delta x_i.$$

The sum $\sum_{i=1}^{n} m_i \cdot \Delta x_i$ is called the lower Riemann sum for f and Δx, and $\sum_{i=1}^{n} M_i \cdot \Delta x_i$ is called the upper Riemann sum or f and Δx. Here "lower" and "upper" refer to the position of M_i and m_i on the y-axis. To get close to your book's notation, we will write $\overline{A}_f(\Delta x)$ and $\underline{A}_f(\Delta x)$ for these sums. Note that unlike the book, we have used Δx instead of n to denote the partition generated by Δx. In our notation, we always have the following fact for any interval $[a, b]$ and any continuous function f on $[a, b]$.

$$\underline{A}_f(\Delta x) \le R_a^b(f, \Delta x) \le \overline{A}_f(\Delta x).$$

EXAMPLE: Suppose we want to estimate the area between the x-axis and the graph of $y = \sin x$ for the interval $[0, \pi]$. Using $\Delta x = \pi/6$, we have the following values for each interval:

i	Interval	$f(x_{i-1})$	m_i	M_i	$M_i - m_i$
1	$[0, \pi/6]$	0	0	$1/2$	$1/2$
2	$[\pi/6, \pi/3]$	$1/2$	$1/2$	$\sqrt{3}/2$	$\frac{\sqrt{3}-1}{2}$
3	$[\pi/3, \pi/2]$	$\sqrt{3}/2$	$\sqrt{3}/2$	1	$1 - \frac{\sqrt{3}}{2}$
4	$[\pi/2, 2\pi/3]$	1	$\sqrt{3}/2$	1	$1 - \frac{\sqrt{3}}{2}$
5	$[2\pi/3, 5\pi/6]$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}/2$	$\frac{\sqrt{3}-1}{2}$
6	$[5\pi/6, \pi]$	$1/2$	0	$1/2$	$1/2$

The lower sum, which is too small, is

$$\underline{A}_f(\Delta x) = \frac{\pi}{6}\left(0 + \frac{1}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{1}{2} + 0\right) = \frac{\pi}{6}\left(1 + \sqrt{3}\right) = 1.43.$$

The upper sum, which is too big, is

$$\overline{A}_f(\Delta x) = \frac{\pi}{6}\left(\frac{1}{2} + \frac{\sqrt{3}}{2} + 1 + 1 + \frac{\sqrt{3}}{2} + \frac{1}{2}\right) = \frac{\pi}{6}\left(3 + \sqrt{3}\right) = 2.48.$$

The Riemann sum, is

$$R_0^{\pi}(\sin x, \frac{\pi}{6}) = \frac{\pi}{6}\left(0 + \frac{1}{2} + \frac{\sqrt{3}}{2} + 1 + \frac{\sqrt{3}}{2} + \frac{1}{2}\right) = \frac{1}{6}\pi\left(2 + \sqrt{3}\right) = 1.954$$

We shall see that the correct answer is 2.

Later we will talk about the maximum value of $M_i - m_i$. In this case it is $1/2$.

The Riemann sum and the area under the curve are trapped between the upper and lower sum. We will see that for a continuous function, all three sums have the same limit as $\Delta x \to 0$. We will call that limit the **Riemann integral** of f on the interval $[a, b]$. We will write $\int_a^b f(x)\,dx$ for that limit.

We are going to want to consider functions that can take negative values. Where the graph of the function we are working with is above the x-axis, we will consider the area between the graph and the x-axis as positive area. Where the graph of the function is below the x-axis, we will consider the area between the graph and the x-axis as negative area.

EXAMPLES: For $f(x) = -x$ on the interval $[-2, 2]$, the area between the graph and the x-axis from $x = -2$ to $x = 0$ is positive area and the area between the graph and the x-axis from $x = 0$ to $x = 2$ is negative area. Therefore, we say that there is 0 area between the graph of $y = -x$ and the x-axis for the interval $[-2, 2]$. Similarly, there is 0 area between the x-axis and the graph of the function $y = \sin x$ for the interval $[0, 2\pi]$. We use Riemann sums to approximate these areas.

We still have

$$\underline{A}_f(\Delta x) \leq R_a^b(f, \Delta x) \leq \overline{A}_f(\Delta x).$$