1. Come to Class

We are now covering material in a way that is not in the book. Come to class.

2. A Tutoring Room is Open

7–9 p.m, Monday, Tuesday, Wednesday, Thursday, Room 140 Lincoln Hall.

3. Exam, Friday October 27, 11 a.m.

On material through antidifferentiation (homework for Thursday).

Section 4 (Liu Qi), Section 5 (Liu Qi) Section 6 (Michael Barrus), Section 8 (Scott Weaver) will take the exam in Room 314 Altgeld Hall.

Section 2 (Isaac Goldbring), Section 7 (Isaac Goldbring), Section 9 (Timothy LeSaulnier) will take the exam in Room 100 MSEB (Materials Science Engineering Building, North-West corner of Green and Mathews.) People in these sections **must** go to this room and not Altgeld Hall to take the exam.

Everyone should by now know their discussion section and section instructor. You will need to enter that on your examination. Bring your U of I identity card to show when turning in the exam.

Review Thursday October 26, Rooms 245, 443, 445 Altgeld Hall, 7-9 p.m.

4. Homework 17 due Tuesday, October 24 at 9 a.m.

Section 4.7: #2, 8, 34, 36.

Section 4.8: #8, 22, 24, 36.

Section 4.9: #6, 10.

5. Homework 18 due Thursday, October 26 at 9 a.m.

Section 5.2:# 12, 14, 16, 18, 24, 36, 40, 42, 44, 48, 58, 60.

6. Written problem for next week

Find the upper sum, the lower sum, the Riemann sum (evaluating at the left of each interval) and the value of $E_f(\Delta x)$ for the function $f(x) = x^4$ on the interval $[-1, 1]$ using $\Delta x = 3/10$. Remember, this is a sample problem for the exam in November.

7. Initial Value Problems.

This is the name for problems where you are given the derivative of a function and a value of the function at one point. You then find the antiderivative (also known as the indefinite integral) of the function you were given and fix the constant so that your answer takes the right value at the given point.

Sample Problem: Solve the initial value problem $\frac{dy}{dx} = (x^2-2x)(x-1)$, $y(1) = 3$.

Ans: $\int (x^2 - 2x)(x - 1)dx$ can be evaluated by setting $u = x^2 - 2x$, so $du = (2x - 2)\,dx$, and $\frac{1}{2}du = (x - 1)dx$. With this substitution, the integral is $\frac{1}{2}\int udu = \frac{1}{4}u^2 + C = \frac{1}{4}(x^2 - 2x)^2 + C$. We want the value 3 when $x = 1$, so $C = 3 - \frac{1}{4}$.

8. More on Area

We are going to want to consider functions that can take negative values. Where the graph of the function we are working with is above the x-axis, we will consider the area between the graph and the x-axis as positive area. Where the graph of the function is below the x-axis, we will consider the area between the graph and the x-axis as negative area.

EXAMPLES: For $f(x) = -x$ on the interval $[-2, 2]$, the area between the graph and the x-axis from $x = -2$ to $x = 0$ is positive area and the area between the graph and the x-axis from $x = 0$ to $x = 2$ is negative area. Therefore, we say that there is 0 area between the graph of $y = -x$ and the x-axis for the interval $[-2, 2]$. Similarly, there is 0 area between the x-axis and the graph of the function $y = \sin x$ for the interval $[0, 2\pi]$. We use Riemann sums to approximate these areas.

We still have

$$\underline{A}_f(\Delta x) \leq R_a^b(f, \Delta x) \leq \overline{A}_f(\Delta x).$$

9. Maximum Change Theorem

To show the above sums all have the same limit, we need the following result. Although it is not in your book, it is the basic fact about continuous functions needed for integration.

Theorem 1 [Maximum Change Theorem]**.** *Let f be a continuous function on a closed interval $[a, b]$. For each positive Δx and the corresponding partition $a = x_0 < x_1 < \cdots < x_n = b$, let M_i denote the maximum value of the function f in the i^{th} interval $[x_{i-1}, x_i]$ and let m_i denote the minimum value of the function f in the i^{th} interval $[x_{i-1}, x_i]$. Set*

$$E_f(\Delta x) = \underset{1 \leq i \leq n}{\text{maximum}} \left(M_i - m_i \right).$$

Then $\lim_{\Delta x \to 0+} E_f(\Delta x) = 0$.

Proof. I will indicate the proof for the case that the derivative f' is bounded. That is, the absolute value $|f'(x)| \leq B$ at all x in $[a, b]$. The most general case, however, must wait for a later course.

Look at any interval $[x_{i-1}, x_i]$, and assume that $M_i = f(\overline{x_i})$ while $m_i = f(\underline{x}_i)$. By the Mean Value Theorem and using absolute values, there is a point c between $\underline{x}_i$ and $\overline{x_i}$ with

$$M_i - m_i = |f'(c) \cdot (\overline{x_i} - \underline{x}_i)| \leq B \cdot \Delta x.$$

Since for the given Δx, $E_f(\Delta x)$ is the maximum of the differences $M_i - m_i$ over all of the intervals $[x_{i-1}, x_i]$, it follows that

$$0 \leq E_f(\Delta x) \leq B \cdot \Delta x.$$

Since $B \cdot \Delta x \to 0$ as $\Delta x \to 0$, it follows from the squeeze law that the same is true for $E_f(\Delta x)$. $\square$

The theorem says that if you look at the maximum change of a continuous function f over the Δx-subintervals of $[a, b]$, that maximum goes to 0 as Δx goes to 0. Note that we can not talk about the change over any particular interval as $\Delta x \to 0$ because as $\Delta x \to 0$, the intervals change. (For later courses, you may want to note that the Maximum Change Theorem is equivalent to the statement that a continuous function on $[a, b]$ is "uniformly continuous" on $[a, b]$.)

Sample Problem: I will let f be a continuous function on a closed interval $[a, b]$. I will define $E_f(\Delta x)$ for you and ask you to state the basic fact about $E_f(\Delta x)$ that is needed for integration. The answer is $\lim_{\Delta x \to 0+} E_f(\Delta x)$.

Example: Let $f(x) = 1 - x^2$ on the interval $[-2, 3]$. Let $\Delta x = 1$. Then we have the following:

$$x_0 = -2,\ x_1 = -1,\ x_2 = 0,\ x_3 = 1,\ x_4 = 2,\ x_5 = 3.$$

We also have the following values for each interval:

i	Interval	$f(x_{i-1})$	m_i	M_i	$M_i - m_i$
1	$[-2, -1]$	-3	-3	0	3
2	$[-1, 0]$	0	0	1	1
3	$[0, 1]$	1	0	1	1
4	$[1, 2]$	0	-3	0	3
5	$[2, 3]$	-3	-8	-3	5

It follows that for this function f, that for $\Delta x = 1$, $E_f(1) = 5$. This is large, but our theorem says that as $\Delta x \to 0$, $E_f(\Delta x) = \max_i(M_i - m_i) \to 0$. Using this table, we can compute the Riemann Sum

$$R_{-2}^{3}(f, 1) = -3 \cdot 1 + 0 \cdot 1 + 1 \cdot 1 + 0 \cdot 1 - 3 \cdot 1 = -5,$$

the lower sum

$$\underline{A}_f(\Delta x) = -3 \cdot 1 + 0 \cdot 1 + 0 \cdot 1 - 3 \cdot 1 - 8 \cdot 1 = -14,$$

and the upper sum

$$\overline{A}_f(\Delta x) = 0 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 0 \cdot 1 - 3 \cdot 1 = -1.$$

Note that $\underline{A}_f(\Delta x) \leq R_{-2}^{3}(f, 1) \leq \overline{A}_f(\Delta x)$ and that $\overline{A}_f(\Delta x) - \underline{A}_f(\Delta x) \leq E_f(1) \cdot (b - a)$ where $a = -2$ and $b = 3$.

10. Equivalence of Integral and Area

Let $A(f)$ be the signed area between the graph of f and the x-axis for the interval $[a,b]$. We are going to show now what the book could not show, namely, that $A(f) = \lim_{\Delta x \to 0} R_a^b(f, \Delta x)$. We assume that $A(f)$ exists as a number; we only want to calculate it.

Theorem 2. *If f is a continuous function on an interval $[a,b]$, then*

$$A(f) = \lim_{\Delta x \to 0+} R_a^b(f, \Delta x).$$

Proof. Fix $\Delta x > 0$, and the corresponding partition $a = x_0 < x_1 < \cdots < x_n = b$ of $[a,b]$. Then $\underline{A}_f(\Delta x) \le A(f) \le \overline{A}_f(\Delta x)$ and $\underline{A}_f(\Delta x) \le R_a^b(f, \Delta x) \le \overline{A}_f(\Delta x)$. We need only show that

$$\lim_{\Delta x \to 0+} \left(\overline{A}_f(\Delta x) - \underline{A}_f(\Delta x) \right) = 0.$$

But

$$\begin{aligned} \overline{A}_f(\Delta x) - \underline{A}_f(\Delta x) &= \sum_{i=1}^n M_i \cdot \Delta x_i - \sum_{i=1}^n m_i \cdot \Delta x_i = \sum_{i=1}^n (M_i - m_i) \cdot \Delta x_i \\ &\le \sum_{i=1}^n E_f(\Delta x) \cdot \Delta x_i = E_f(\Delta x) \cdot \sum_{i=1}^n \Delta x_i = E_f(\Delta x) \cdot (b-a), \end{aligned}$$

and $\lim_{\Delta x \to 0+} E_f(\Delta x) \cdot (b-a) = 0$, so the result follows by the squeeze law. □

We have shown that the limit of the Riemann sums for a continuous function f exists and equals the signed area between the graph and the x-axis. We will write $\int_a^b f(x)\,dx$ for this limit. That is we have the following definition.

Definition 3. *Given a continuous function f on a closed interval $[a,b]$,*

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0+} R_a^b(f, \Delta x).$$

We call $\int_a^b f(x)\,dx$ the (definite) ***integral*** *or* ***Riemann integral*** *of the function f from a to b. We call the function f the* ***integrand*** *for this integral, and the numbers a and b are called, respectively, the* ***lower and upper limits of integration****.*

11. Basic Properties of the integral

Here are some basic properties of the integral you will need to know. An operation such as taking the integral is called **linear** if it preserves addition and multiplication by constants.

Theorem 4 [Linearity of the Integral]**.** *If f and g are continuous functions on $[a,b]$ and c is a constant, then*

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

and

$$\int_a^b c\cdot f(x)\,dx = c\cdot\int_a^b f(x)\,dx.$$

Proof. To show this, we take the limit as $\Delta x \to 0+$ of both sides of the following two equations.

$$\begin{aligned} R_a^b(f,\Delta x) + R_a^b(g,\Delta x) &= \sum_{i=1}^n f(x_{i-1})\cdot\Delta x_i + \sum_{i=1}^n g(x_{i-1})\cdot\Delta x_i \\ &= \sum_{i=1}^n (f(x_{i-1})+g(x_{i-1}))\cdot\Delta x_i = R_a^b(f+g,\Delta x) \end{aligned}$$

and

$$c\cdot R_a^b(f,\Delta x) = c\cdot\left(\sum_{i=1}^n f(x_{i-1})\cdot\Delta x_i\right) = \sum_{i=1}^n (c\cdot f(x_{i-1}))\,\Delta x_i = R_a^b(c\cdot f,\Delta x).$$

EXAMPLES: $\int_{-5}^7 (3\sin x + 4x^2)\,dx = 3\int_{-5}^7 \sin x\,dx + 4\int_{-5}^7 x^2 dx$.

The following result is clear if you remember that the integral is the signed area between the x-axis and the graph of the function.

Theorem 5. *If f is continuous on $[a,b]$, and $m \le f(x) \le M$ for all x in $[a,b]$, then*

$$m\cdot(b-a) \le \int_a^b f(x)\,dx \le M\cdot(b-a).$$

Proof. The result follows from the fact that for each $\Delta x > 0$,

$$m\cdot(b-a) = \sum_{i=1}^n m\cdot\Delta x_i \le R_a^b(f,\Delta x) = \sum_{i=1}^n f(x_{i-1})\cdot\Delta x_i \le \sum_{i=1}^m M\cdot\Delta x_i = M\cdot(b-a). \quad \square$$

EXAMPLE:

$$0 = 0\cdot\pi \le \int_0^\pi \sin x\,dx \le 1\cdot\pi = \pi.$$