1. Come to Class

We are now covering material in a way that is not in the book. Come to class.

2. A Tutoring Room is Open

7–9 p.m, Monday, Tuesday, Wednesday, Thursday, Room 140 Lincoln Hall.

3. Exam, Friday October 27, 11 a.m.

On material through antidifferentiation (homework for Thursday).

Section 4 (Liu Qi), Section 5 (Liu Qi) Section 6 (Michael Barrus), Section 8 (Scott Weaver) will take the exam in Room 314 Altgeld Hall.

Section 2 (Isaac Goldbring), Section 7 (Isaac Goldbring), Section 9 (Timothy LeSaulnier) will take the exam in Room 100 MSEB (Materials Science Engineering Building, North-West corner of Green and Mathews.) People in these sections **must** go to this room and not Altgeld Hall to take the exam.

Everyone should by now know their discussion section and section instructor. You will need to enter that on your examination. Bring your U of I identity card to show when turning in the exam.

Review Thursday October 26, Rooms 245, 443, 445 Altgeld Hall, 7-9 p.m.

4. Homework 17 due Tuesday, October 24 at 9 a.m.

Section 4.7: #2, 8, 34, 36.

Section 4.8: #8, 22, 24, 36.

Section 4.9: #6, 10.

5. Homework 18 due Thursday, October 26 at 9 a.m.

Section 5.2:# 12, 14, 16, 18, 24, 36, 40, 42, 44, 48, 58, 60.

6. Written problem for this week

Find the upper sum, the lower sum, the Riemann sum (evaluating at the left of each interval) and the value of $E_f(\Delta x)$ for the function $f(x) = x^4$ on the interval $[-1, 1]$ using $\Delta x = 3/10$. Remember, this is a sample problem for the exam in November.

7. The Riemann Integral

We have shown that the **limit** of the Riemann sums for a continuous function f exists and equals the signed area between the graph and the x-axis. We write $\int_a^b f(x)\,dx$ for this limit. That is we have the following definition.

Definition 1. *Given a continuous function f on a closed interval $[a, b]$,*

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0+} R_a^b(f, \Delta x).$$

We call $\int_a^b f(x)\,dx$ the (definite) ***integral*** *or* ***Riemann integral*** *of the function f from a to b. We call the function f the* ***integrand*** *for this integral, and the numbers a and b are called, respectively, the* ***lower and upper limits of integration****.*

8. Limits of Integration

The following result is clear if you remember that the integral is the signed area between the x-axis and the graph of the function.

Theorem 2. *If f is continuous on $[a,b]$ and $a < c < b$, then*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Proof. You can prove this by considering Riemann sums with $\Delta x = \frac{c-a}{n}$ for positive integers n. □

EXAMPLE:

$$\int_{-2}^2 x^2\,dx = \int_{-2}^0 x^2 dx + \int_0^2 x^2 dx.$$

There are times when one needs to consider limits of integration that are not in the usual order.

Definition 3. *If $a < b$, and f is continuous on $[a,b]$, we set*

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

Also, we set

$$\int_a^a f(x)\,dx = 0.$$

This means that we have the following no matter where a, b, and c are:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

The variable used for the integral is like the variable for summation; changing it does not change the answer.

EXAMPLE: $\int_0^1 x^2 dx = \int_0^1 t^2 dt$

Sample Problem. Evaluate $\int_2^5 e^{-t^2} dt + \int_5^2 e^{-x^2} dx$. **Ans:** 0.

Here is one more property of the integral: The bigger the function you are integrating, the bigger the integral. That is, if $f \le g$ on $[a,b]$, then $\int_a^b f(x)dx \le \int_a^b g(x)dx$. This is clearly true for Riemann sums, so it is true for their limits.

EXAMPLE: $\int_1^2 x dx \le \int_1^2 x^2\ dx$.

9. Upper and lower sums

We have seen that for any positive Δx,

$$\underline{A}_f(\Delta x) \leq \int_a^b f(x)\,dx \leq \overline{A}_f(\Delta x).$$

Here is a principle that often works (but it does not always work) to show that integrating a particular continuous function f answers a problem.

Theorem 4. *Let S be a real number (for example, signed area) and let f a continuous function on a closed and bounded interval $[a, b]$. Suppose that for each positive Δx, we know that $\underline{A}_f(\Delta x) \leq S \leq \overline{A}_f(\Delta x)$. Then $S = \int_a^b f(x)\,dx$.*

Proof. We know that $\underline{A}_f(\Delta x) \leq \int_a^b f(x)\,dx \leq \overline{A}_f(\Delta x)$. Since also $\underline{A}_f(\Delta x) \leq S \leq \overline{A}_f(\Delta x)$ and $\overline{A}_f(\Delta x) - \underline{A}_f(\Delta x)$ has limit 0 as $\Delta x \to 0$, It follows from the squeeze law that $S = \int_a^b f(x)\,dx$. □

EXAMPLE. The signed area between the graph of a continuous function and the part of the x-axis from a to b is an example of such a quantity S.

10. Error Sum Principle

There is a principle that you can use throughout your career whenever you need to use an integral to measure some quantity. This principle tells you when you have set up the correct integral to make the desired measurement. It is more general than the principle using upper and lower sums given by the last theorem. When you cut up a quantity S into pieces S_i and use a term of the Riemann sum to approximate S_i you have an error e_i.

It is not enough that each error goes to 0; it is the sum of the errors that must go to 0. For example, if you divide the interval over which you are integrating into n pieces, and each error is $1/n$, then each error will go to 0, but the sum of the errors will always be 1.

Here is a principle that tells you when the sum of these errors goes to 0 as Δx goes to 0. It says that each error should be smaller in absolute value than Δx multiplied by a function $E(\Delta x)$, where $\lim_{\Delta x \to 0+} E(\Delta x) = 0$. Often the function $E_f(\Delta x)$ that we have discussed before or a sum of such functions works for the function E.

Theorem 5 [Error-Sum Principle]**.** *Let S be a number and let f be a continuous function of x on an interval $[a, b]$. You know that $S = \int_a^b f(x)\,dx$ if you can find a function $E(\Delta x)$ of Δx that has limit 0 as $\Delta x \to 0$ and that also has the following property: For each positive Δx and corresponding partition $a = x_0 < x_1 < \cdots <$*

$x_n = b$ of $[a, b]$, you can write S as a sum $S = \sum_{i=1}^{n} S_i$ in such a way that for each i between 1 and n,

$$|S_i - f(x_{i-1}) \cdot \Delta x_i| \leq E(\Delta x) \cdot \Delta x_i.$$

Example: In finding the area under a curve, we saw that the absolute value of the difference between the area A_i for the interval $[x_{i-1}, x_i]$ and the term $f(x_{i-1}) \cdot \Delta x_i$ of the Riemann sum is at most $M_i \cdot \Delta x_i - m_i \cdot \Delta x_i \leq E_f(\Delta x) \cdot \Delta x_i$.

Here is the **proof** of the Error-Sum Principle:

$$\begin{aligned} \left|S - R_a^b(f, \Delta x)\right| &= \left|\sum_{i=1}^{n} S_i - \sum_{i=1}^{n} f(x_{i-1})\Delta x_i\right| = \left|\sum_{i=1}^{n} [S_i - f(x_{i-1})\Delta x_i]\right| \\ &\leq \sum_{i=1}^{n} |S_i - f(x_{i-1})\Delta x_i| \leq \sum_{i=1}^{n} E(\Delta x) \cdot \Delta x_i = E(\Delta x) \cdot (b - a). \end{aligned}$$

Since $E(\Delta x) \cdot (b - a)$ has limit 0 as $\Delta x \to 0$, the same is true for $\left|S - R_a^b(f, \Delta x)\right|$. That is,

$$S = \lim_{\Delta x \to 0+} R_a^b(f, \Delta x) = \int_a^b f(x)dx. \ \square$$

11. Other sums with limit the integral

Given a continuous function f on $[a, b]$ and $\Delta x > 0$, we have defined the Riemann sum of a function using evaluation at the left of each partition interval $[x_{i-1}, x_i]$. That is, $R_a^b(f, \Delta x) = \sum_{i=1}^{n} f(x_{i-1})\Delta x_i$. If we evaluate at any other point of the interval $[x_{i-1}, x_i]$, we still have a sum between the lower sum and the upper sum. Since the difference between the upper sum and the lower sum for Δx has limit 0 as $\Delta x \to 0$, we have the following result.

Theorem 6. *Let f be a continuous function on a closed interval $[a, b]$. For each $\Delta x > 0$ and corresponding partition $a = x_0 < x_1 < \cdots < x_n = b$, let c_i be a point in the interval $[x_{i-1}, x_i]$. Then*

$$\int_a^b f(x)dx = \lim_{\Delta x \to 0+} R_a^b(f, \Delta x) = \lim_{\Delta x \to 0+} \sum_{i=1}^{n} f(x_{i-1})\Delta x_i = \lim_{\Delta x \to 0+} \sum_{i=1}^{n} f(c_i)\Delta x_i.$$

12. Fundamental Theorem of Calculus

We can use the last result to establish the validity of the method for exactly evaluating integrals. This method is called the Fundamental Theorem of Calculus. It says that to evaluate an integral, all you have to do is find an antiderivative of the integrand

and take the change of that antiderivative on the interval. Your book also calls this the **Evaluation Theorem**.

Sample Problem. Evaluate $\int_2^3 2x dx$. **Ans.** Since $D_x x^2 = 2x$, the answer is $3^2 - 2^2 = 5$.

The reason the method of the Evaluation Theorem works is as follows. If $Y = F(x)$ is an antiderivative of f, then $F(b) - F(a) = \sum \Delta Y_i$, the sum of the changes of F over the intervals of the partition for Δx. We have seen that for each i, ΔY_i is close to the differential $dY_i = f(x_{i-1})\Delta x$, which is the i^{th} term of the Riemann sum. The question is, is it close enough. The answer is **yes** because f is continuous.

Theorem 7 [Fundamental Theorem of Calculus]**.** *Let f be a continuous function on an interval $[a, b]$, and assume that F is an* ***antiderivative*** *of f on $[a, b]$. Then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof. For each positive Δx, the overall change of $Y = F(x)$ on the interval $[a, b]$ is the sum of its changes over the subintervals. That is,

$$F(b) - F(a) = \sum_{i=1}^{n} \Delta Y_i = \sum_{i=1}^{n} \left(F(x_i) - F(x_{i-1})\right).$$

Since $F' = f$, by the Mean Value Theorem, for each i, there is a point c_i in (x_{i-1}, x_i) such that $\Delta Y_i = F(x_i) - F(x_{i-1}) = f(c_i) \cdot \Delta x_i$. Therefore,

$$\begin{aligned}\int_a^b f(x)dx &= \lim_{\Delta x \to 0+} \sum_{i=1}^{n} f(x_{i-1})\Delta x_i = \lim_{\Delta x \to 0+} \sum_{i=1}^{n} f(c_i)\Delta x_i \\ &= \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) = F(b) - F(a). \ \square\end{aligned}$$

We will soon give a quite different proof of the Fundamental Theorem of Calculus.

EXAMPLES: Since $\frac{1}{3}x^3$ is an antiderivative of x^2, the area under the graph of $y = x^2$ for the interval $[1, 3]$ is given by

$$\int_1^3 x^2 dx = \left[\frac{1}{3}x^3\right]_1^3 = \frac{1}{3}(3)^3 - \frac{1}{3}1^3 = 8.6666.$$

Notice the use of the notation $[\cdot]_1^3$ to indicate evaluation at the upper limit of integration minus evaluation at the lower limit of integration. In the literature you often see only this] and not $[..]_1^3$.

If we took the antiderivative to be $\frac{1}{3}x^3 + C$ for some constant C, we would be adding and then subtracting C, so it would not effect the final answer. Here is another example:

Example: $\int_0^\pi \sin x dx = [-\cos x]_0^\pi = -\cos\pi - (-\cos 0) = 1 + 1 = 2.$

Example:$\int_1^{e^2} \frac{1}{x} dx = [\ln x]_1^{e^2} = \ln e^2 - \ln 1 = 2.$

If we want to evaluate $\int_a^b f(x)dx$, the antiderivative sometimes changes in the interval. Another possibility is the function may not be continuous at some point or points in the interval but is continuous from the right and left of those points. In either case, we need to break up the integral. Here is an example: $\int_{-1}^1 x dx = \left[\frac{1}{2}x^2\right]_{-1}^1 = 0$, but

$$\int_{-1}^1 |x|\, dx = \int_{-1}^0 -x dx + \int_0^1 x dx = \left[\frac{-1}{2}x^2\right]_{-1}^0 + \left[\frac{1}{2}x^2\right]_0^1 = 1.$$

Here is another example: We want to integrate f from -2 to 3 where $f(x) = x^2$ for $x \leq 1$ and $f(x) = x^2 + 2$ for $x > 1$. Then

$$\begin{aligned}\int_{-2}^3 f(x)\, dx &= \int_{-2}^1 x^2 dx + \int_1^3 \left(x^2 + 2\right)\, dx = \left[\frac{1}{3}x^3\right]_{-2}^1 + \left[\frac{1}{3}x^3 + 2x\right]_1^3 \\ &= \frac{1}{3} - \frac{-8}{3} + \frac{27}{3} + 6 - \frac{1}{3} - 2 = \frac{47}{3}.\end{aligned}$$

If the antiderivative changes at a point, integrate to that point, and add as a second integral the integral beyond that point.

Example: $\int_{-1}^1 |x|\, dx = \int_{-1}^0 -x dx + \int_0^1 x dx = [\frac{-1}{2}x^2]_{-1}^0 + [\frac{1}{2}x^2]_0^1.$