1. Come to Class

We are now covering material in a way that is not in the book. Come to class.

2. A Tutoring Room is Open

7–9 p.m, Monday, Tuesday, Wednesday, Thursday, Room 140 Lincoln Hall.

3. Exam, Friday October 27, 11 a.m.

On material through antidifferentiation (homework for Thursday).

Section 4 (Liu Qi), Section 5 (Liu Qi) Section 6 (Michael Barrus), Section 8 (Scott Weaver) will take the exam in Room 314 Altgeld Hall.

Section 2 (Isaac Goldbring), Section 7 (Isaac Goldbring), Section 9 (Timothy LeSaulnier) will take the exam in Room 100 MSEB (Materials Science Engineering Building, North-West corner of Green and Mathews.) People in these sections **must** go to this room and not Altgeld Hall to take the exam.

Everyone should by now know their discussion section and section instructor. You will need to enter that on your examination. Bring your U of I identity card to show when turning in the exam.

Review Thursday October 26, Rooms 245, 443, 445 Altgeld Hall, 7-9 p.m.

4. Homework 18 due Thursday, October 26 at 9 a.m.

Section 5.2:# 12, 14, 16, 18, 24, 36, 40, 42, 44, 48, 58, 60.

5. Homework 19 due Tuesday, October 31 at 9 a.m.

Section 5.3: #20, 22, 36, 38, 42. Here, L_n is the same as $\underline{A}_f(\Delta x)$ and U_n is the same as $\overline{A}_f(\Delta x)$, where $\Delta x = (b-a)/n$ for the given value of n.

Section 5.4: #2, 8, 22, 26, 28. For these problems, $\Delta x = (b-a)/n$ for the given natural number n, and x_i^* is the left endpoint x_{i-1} of each interval $[x_{i-1}, x_i]$.

6. Written problem for next week

Find the upper sum, the lower sum and the Riemann sum (evaluating at the left of each interval) for the function $f(x) = \cos x$ on the interval $[-\frac{\pi}{2}, \frac{\pi}{4}]$ using $\Delta x = \frac{\pi}{6}$. Also find $E_f(\Delta x)$ for this example.

7. The Riemann sum as a sum of differentials

Let $y = f(x)$, where f is continuous on the interval $[a, b]$. Suppose $Y = F(x)$ is an antiderivative of f. We have seen that the integral of f is given by then $F(b) - F(a)$. Given $\Delta x > 0$, We can look at the sum of the changes of F over the intervals of the corresponding partition. Let $\Delta F_i = F(x_i) - F(x_{i-1})$ be the change of F for the interval $[x_{i-1}, x_i]$, and let dF_i be the differential $f(x_{i-1})\Delta x_i$ for the interval $[x_{i-1}, x_i]$, then

$$\int_a^b f(x)dx = F(b) - F(a) = \sum \Delta F_i = \lim_{\Delta x \to 0} \sum f(x_{i-1})\Delta x_i = \lim_{\Delta x \to 0} \sum dF_i.$$

What is going on here is that for each i, ΔF_i is close to the differential $dF_i = f(x_{i-1})\Delta x_i$. Indeed, by the Mean Value Theorem, for some point c_i in the interval $[x_{i-1}, x_i]$, $\Delta F_i = f(c_i)\Delta x_i$, so the absolute value of the error that occurs in replacing ΔF_i with dF_i is at most $(M_i - m_i)\,\Delta x_i \leq E_f\,(\Delta x)\,\Delta x_i$. This means that the sum of these errors is at most $E_f\,(\Delta x)\,(b-a)$, and this goes to 0 as $\Delta x \to 0$. The fact that the integral is the limit of a sum of differentials is indicated by the equation $\int_a^b f(x)dx = \int_a^b dF$. This is also why I have chosen to evaluate Riemann sums at the left of each interval.

8. The integral as an average

Let $y = f(x)$, f continuous on the interval $[a, b]$. Given $\Delta x > 0$, the positive weights $\Delta x_i/(b-a)$ add up to 1, so it makes sense to call $R_a^b(f, \Delta x)/(b-a)$ an average of the values of $f(x_{i-1})$ for $1 \leq i \leq n$. We call the limit

$$y_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

the average value of f on $[a, b]$. If m is the minimum value of $f(x)$ and M the maximum in the interval $[a, b]$, then

$$m \cdot (b-a) \leq \int_a^b f(x)dx \leq M \cdot (b-a),$$

so $m \leq y_{avg} \leq M$. By the Intermediate Value Theorem, there is a point c in $[a, b]$ with

$$y_{avg} = f(c).$$

The fact of the existence of c is called the **Mean Value Theorem of Integral Calculus.**

Sample Problem Let $f(x) = x^2$. Find the average value of f on the interval $[0, 3]$ and the point where f takes that average value. **Ans.**

$$y_{avg} = \frac{1}{3}\int_0^3 x^2 = \frac{1}{9}3^3 = 3$$

is the average value of f on $[0, 3]$, and the point where f takes that average value is $c = \sqrt{3}$; that is, $y_{avg} = f(\sqrt{3})$.

9. Existence of an antiderivative

Let f be continuous on $[a, b]$. The Fundamental Theorem of Calculus says that to evaluate the definite integral $\int_a^b f(x)\,dx$, all we need to do is find some antiderivative F of f and calculate $F(b) - F(a)$. We now show that even though we may not be able to guess what it is, there is always an antiderivative of f namely

$$F(x) = \int_a^x f(t)\,dt \text{ where } a \leq x \leq b.$$

Notice here, we have changed the variable of integration to t. It is clear that $F(a) = 0$, and of course $F(b) = \int_a^b f(t)\,dt$. Graphically, it seems clear that $F' = f$.

To see by calculations that $F' = f$ on $[a, b]$, fix x with $a \leq x < b$. If $\Delta x > 0$ and $x + \Delta x \leq b$, then

$$\frac{F(x+\Delta x) - F(x)}{\Delta x} = \frac{\int_x^{x+\Delta x} f(t)\,dt}{\Delta x} = f(c_{\Delta x})$$

for some point $c_{\Delta x}$ in the interval $[x, x + \Delta x]$. This is by the Mean-Value Theorem of Integral Calculus. Since f is continuous, and $c_{\Delta x}$ is in $[x, x + \Delta x]$,

$$\lim_{\Delta x \to 0+} \frac{F(x+\Delta x) - F(x)}{\Delta x} = \lim_{\Delta x \to 0+} f(c_{\Delta x}) = f(x).$$

On the other hand, if $a < x \leq b$ and $\Delta x < 0$ but still $a < x + \Delta x$, then since

$$\int_a^x f(t)dt + \int_x^{x+\Delta x} f(t)dt = \int_a^{x+\Delta x} f(t)dt$$

$$\begin{aligned} \frac{F(x+\Delta x) - F(x)}{\Delta x} &= \frac{\int_a^{x+\Delta x} f(t)\,dt - \int_a^x f(t)dt}{\Delta x} \\ &= \frac{\int_x^{x+\Delta x} f(t)dt}{\Delta x} = \frac{\int_{x+\Delta x}^x f(t)dt}{|\Delta x|} = f(c_{\Delta x}) \end{aligned}$$

for some point $c_{\Delta x}$ in $[x+\Delta x, x]$. Again, it follows that $\lim_{\Delta x \to 0-} \frac{F(x+\Delta x)-F(x)}{\Delta x} = f(x)$. Therefore, $F'(x) = f(x)$ for all x in $[a, b]$, with the limit being one sided at the end points.

EXAMPLES:

$$D_x \int_1^x t^3\,dt = x^3.$$

$$D_x \int_0^x \sin t\,dt = \sin x.$$

$$D_x \int_a^{g(x)} f(t)\,dt = f(g(x)) \cdot g'(x).$$

$$D_x \int_1^{\sin x} t^3\,dt = \sin^3 x \cdot \cos x.$$

$$\begin{aligned} D_x \int_{x^2}^{x^3} e^t dt &= D_x \int_{x^2}^{1} e^t dt + D_x \int_1^{x^3} e^t dt \\ &= D_x \int_1^{x^3} e^t dt - D_x \int_1^{x^2} e^t dt \\ &= e^{x^3} \cdot 3x^2 - e^{x^2} \cdot 2x. \end{aligned}$$

Sample Problem. Write a formula for a function of x on the real line for which the derivative is e^{-x^2} and the function takes the value 0 at $x = 0$. **Ans.** $\int_0^x e^{-t^2} dt$.

10. Another Proof of the Fundamental Theorem of Calculus

By what we have shown here, if G is any antiderivative of f, then $G(x)$ differs from the antiderivative $\int_a^x f(t)\,dt$ by a constant. Since the function $G(x) - G(a)$ is 0 at $x = a$, we have $G(x) - G(a) = \int_a^x f(t)\,dt$ for all x in $[a, b]$. In particular,

$$G(b) - G(a) = \int_a^b f(t)\,dt.$$