

1. A TUTORING ROOM IS OPEN

7–9 p.m, Monday, Tuesday, Wednesday, Thursday, Room 140 Lincoln Hall.

2. HOMEWORK 19 DUE TUESDAY, OCTOBER 31 AT 9 A.M.

Section 5.3: #20, 22, 36, 38, 42. Here, L_n is the same as $\underline{A}_f(\Delta x)$ and U_n is the same as $\overline{A}_f(\Delta x)$, where $\Delta x = (b - a)/n$ for the given value of n .

Section 5.4: #2, 8, 22, 26, 28. For these problems, $\Delta x = (b - a)/n$ for the given natural number n , and x_i^* is the left endpoint x_{i-1} of each interval $[x_{i-1}, x_i]$.

3. HOMEWORK 20 DUE THURSDAY, NOVEMBER 2 AT 9 A.M.

Section 5.5: #6, 10, 14, 22, 24, 28, 30, 34, 36, 44.

4. WRITTEN PROBLEM FOR THIS WEEK

Find the upper sum, the lower sum and the Riemann sum (evaluating at the left of each interval) for the function $f(x) = \cos x$ on the interval $[-\frac{\pi}{2}, \frac{\pi}{4}]$ using $\Delta x = \frac{\pi}{6}$. Also find $E_f(\Delta x)$ for this example.

5. GRADE ON EXAM 3

≤ 34 , F, 16. [35, 39], D-, 6. [40, 44], D, 13. [45, 49], D+, 17. [50, 54], C-, 10. [55, 59], C, 10. [60, 64], C+, 10. [65, 69], B-, 16. [70, 74], B, 14. [75, 79], B+, 14. [80, 84], A-, 17. [85, 94], A, 20. [95, 100], A+, 4.

6. ANSWERS TO EXAM

1 a) Find $\int x^6 + 4x^3 - 3x + 2 \, dx = \frac{1}{7}x^7 + x^4 - \frac{3}{2}x^2 + 2x + C$.

b) Find $\int \sin(x^3 + 6x + 1)(x^2 + 2) \, dx$. Ans:

Let $u = x^3 + 6x + 1$, so $du = (3x^2 + 6) \, dx$, so $1/3 \, du = (x^2 + 2) \, dx$. Then

$$\begin{aligned} \int \sin(x^3 + 6x + 1)(x^2 + 2) \, dx &= \frac{1}{3} \int \sin u \, du = -\frac{1}{3} \cos u + C \\ &= -\frac{1}{3} \cos(x^3 + 6x + 1) + C. \end{aligned}$$

1 c) Solve the initial value problem $\frac{dy}{dx} = 1/x$, $y(e) = 3$. Ans. $y = \ln x + C$. Since $\ln e = 1$, $C = 2$, so $y = \ln x + 2$.

2) Given a differentiable function $y = f(x)$, you want to find a value x such that $f(x) = 0$. You will use Newton's rule. You make a guess x_1 such that the derivative $f'(x_1) \neq 0$.

a) Write the formula for the tangent line L at the point $(x_1, f(x_1))$. Ans: $y - f(x_1) = f'(x_1)(x - x_1)$.

b) Your next guess x_2 is the x -coordinate of the point where that tangent line L intersects what line? Ans. The x -axis.

3 a) Complete the following: For $y = x^{2/3}$, the differential $dy = \dots$ Ans. $dy = \frac{2}{3}x^{-1/3}dx$

3 b) Use the answer to Part a and the fact that $27^{2/3} = 9$ to estimate $25^{2/3} - 9$. Ans.

$$dy = \frac{2}{3}(27)^{-1/3}(-2) = \frac{-4}{9} = -0.44444. \quad \text{Note: } 25^{2/3} - 9 = -0.45012$$

4a) $\lim_{x \rightarrow 0} \frac{3 \tan 5x}{2x} = ?$ Ans:

$$\lim_{x \rightarrow 0} \frac{3 \tan 5x}{2x} = \lim_{x \rightarrow 0} \frac{D_x(3 \tan 5x)}{D_x(2x)} = \lim_{x \rightarrow 0} \frac{15 \sec^2 5x}{2} = \frac{15}{2}.$$

4b) $\lim_{x \rightarrow +\infty} \frac{4x^2 - 2x + 8}{3x^2 - 3x + 4} = ?$ Ans: $4/3$.

5) Let y be a function of x satisfying the equation $4 \sin y = x^3$. Implicitly differentiate this equation twice to obtain an equation involving x , y , dy/dx , and d^2y/dx^2 .
Ans:

$$\begin{aligned} 4 \sin y &= x^3 \\ 4 \cos y \frac{dy}{dx} &= 3x^2 \\ -4 \sin y \left(\frac{dy}{dx} \right)^2 + 4 \cos y \frac{d^2y}{dx^2} &= 6x. \end{aligned}$$

6) A function f has a second derivative everywhere on the real line. In each of the following cases, what can you say about f and its graph over an open interval (a, b) if you are given the following information?

Case 1) The first derivative $f'(x) = 0$ for each x in (a, b) . The function is constant.

Case 2) The first derivative $f'(x) < 0$ for each x in (a, b) . The function is decreasing

Case 3) The second derivative $f''(x) > 0$ for each x in (a, b) . The graph is concave up.

Case 4) The second derivative $f''(x)$ changes sign at a point $x = c \in (a, b)$. The point c is an inflection point where the concavity changes.

7) Suppose f is a differentiable function on the real line and the derivative $f'(x) \leq 2$ for all x . Is it possible that $f(1) = 4$ and $f(3) = 10$? **Explain.** Ans. No because

$$\frac{f(3) - f(1)}{3 - 1} = 3,$$

and $f'(x)$ is never 3, so by the Mean Value Theorem, this can never happen.

8) A ball is tossed up from the edge of a 192 foot (that's $16 \cdot 12$ foot) building with an initial velocity of 16 feet per second. It goes up and then falls to ground level and continues falling into a narrow pipe that opens at ground level. A camera is mounted at ground level 16 feet from the opening of the pipe. Let ground level be height $y = 0$. Remember, the acceleration due to gravity is $a = -32$ feet/sec².

a) Find a formula for the height y of the ball at any time t in seconds after the ball is tossed up. Ans.

$$\begin{aligned} a(t) &= -32 \\ v(t) &= -32t + 16 \\ y(t) &= -16t^2 + 16t + 192. \end{aligned}$$

b) When will the ball reach its maximum height? Ans. When $v = 0$ at $t = 1/2$ second.

c) When will the ball enter the pipe at ground level? Ans: $y(t) = 0$ when $t^2 - t - 12 = (t - 4)(t + 3) = 0$, and the positive time when this happens is at $t = 4$ seconds.

d) Find how fast the angle θ between ground level and the line from the camera to the ball is changing when the ball enters the pipe, that is, when $\theta = 0$. Show all your work. Ans.

$$\begin{aligned} \tan \theta &= \frac{y(t)}{16} = -t^2 + t + 12 \\ \sec^2 \theta \frac{d\theta}{dt} &= -2t + 1 \end{aligned}$$

At $t = 4$, $\theta = 0$, $\sec^2 \theta = 1$, and $\frac{d\theta}{dt} = -7$ radians per second.

7. EXISTENCE OF AN ANTIDERIVATIVE

Let f be continuous on $[a, b]$. The Fundamental Theorem of Calculus says that to evaluate the definite integral $\int_a^b f(x) dx$, all we need to do is find some antiderivative F of f and calculate $F(b) - F(a)$. We now show that even though we may not be able to guess what it is, there is always an antiderivative of f namely

$$F(x) = \int_a^x f(t) dt \text{ where } a \leq x \leq b.$$

Notice here, we have changed the variable of integration to t . It is clear that $F(a) = 0$, and of course $F(b) = \int_a^b f(t) dt$. Graphically, it seems clear that $F' = f$.

To see by calculations that $F' = f$ on $[a, b]$, fix x with $a \leq x < b$. If $\Delta x > 0$ and $x + \Delta x \leq b$, then

$$\frac{F(x + \Delta x) - F(x)}{\Delta x} = \frac{\int_x^{x+\Delta x} f(t) dt}{\Delta x} = f(c_{\Delta x})$$

for some point $c_{\Delta x}$ in the interval $[x, x + \Delta x]$. This is by the Mean-Value Theorem of Integral Calculus. Since f is continuous, and $c_{\Delta x}$ is in $[x, x + \Delta x]$,

$$\lim_{\Delta x \rightarrow 0^+} \frac{F(x + \Delta x) - F(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0^+} f(c_{\Delta x}) = f(x).$$

On the other hand, if $a < x \leq b$ and $\Delta x < 0$ but still $a < x + \Delta x$, then since

$$\int_a^x f(t) dt + \int_x^{x+\Delta x} f(t) dt = \int_a^{x+\Delta x} f(t) dt$$

$$\begin{aligned} \frac{F(x + \Delta x) - F(x)}{\Delta x} &= \frac{\int_a^{x+\Delta x} f(t) dt - \int_a^x f(t) dt}{\Delta x} \\ &= \frac{\int_x^{x+\Delta x} f(t) dt}{\Delta x} = \frac{\int_{x+\Delta x}^x f(t) dt}{|\Delta x|} = f(c_{\Delta x}) \end{aligned}$$

for some point $c_{\Delta x}$ in $[x + \Delta x, x]$. Again, it follows that $\lim_{\Delta x \rightarrow 0^-} \frac{F(x + \Delta x) - F(x)}{\Delta x} = f(x)$. Therefore, $F'(x) = f(x)$ for all x in $[a, b]$, with the limit being one sided at the end points.

EXAMPLES:

$$D_x \int_1^x t^3 dt = x^3.$$

$$D_x \int_0^x \sin t dt = \sin x.$$

$$D_x \int_a^{g(x)} f(t) dt = f(g(x)) \cdot g'(x).$$

$$D_x \int_1^{\sin x} t^3 dt = \sin^3 x \cdot \cos x.$$

$$\begin{aligned} D_x \int_{x^2}^{x^3} e^t dt &= D_x \int_{x^2}^1 e^t dt + D_x \int_1^{x^3} e^t dt \\ &= D_x \int_1^{x^3} e^t dt - D_x \int_1^{x^2} e^t dt \\ &= e^{x^3} \cdot 3x^2 - e^{x^2} \cdot 2x. \end{aligned}$$

Sample Problem. Write a formula for a function of x on the real line for which the derivative is e^{-x^2} and the function takes the value 0 at $x = 0$. **Ans.** $\int_0^x e^{-t^2} dt$.

8. ANOTHER PROOF OF THE FUNDAMENTAL THEOREM OF CALCULUS

By what we have shown here, if G is any antiderivative of f , then $G(x)$ differs from the antiderivative $\int_a^x f(t) dt$ by a constant. Since the function $G(x) - G(a)$ is 0 at $x = a$, we have $G(x) - G(a) = \int_a^x f(t) dt$ for all x in $[a, b]$. In particular,

$$G(b) - G(a) = \int_a^b f(t) dt.$$

9. CHANGING LIMITS OF INTEGRATION AFTER A SUBSTITUTION

Suppose that $Y = F(x)$ on an interval $[a, b]$, and $\frac{dY}{dx} = F'(x) = f(x)$ on $[a, b]$, then as x goes from a to b , $Y = F(x)$ goes from $F(a)$ to $F(b)$. We don't know which of the numbers $F(a)$ and $F(b)$ is bigger. The Fundamental Theorem of Calculus says that

$$F(b) - F(a) = \int_{F(a)}^{F(b)} dY = \int_a^b f(x) dx.$$

That is, when we make the substitution $Y = F(x)$, so that $dY = f(x) dx$, we can change the dx integral with limits of integration a and b to the corresponding dY integral; we must then change the limits of integration to $F(a)$ and $F(b)$.

Suppose now that $Y = F(u)$, that $F'(u) = f(u)$, and that $u = g(x)$, so that

$$dY = F'(u) du = f(u) du = f(g(x)) g'(x) dx.$$

Again, we assume that the interval for the variable x is $[a, b]$. As x goes from a to b , $u = g(x)$ goes from $g(a)$ to $g(b)$, and Y goes from $F(g(a))$ to $F(g(b))$. Therefore,

$$F(g(b)) - F(g(a)) = \int_{F(g(a))}^{F(g(b))} dY = \int_{g(a)}^{g(b)} f(u) du = \int_a^b f(g(x)) g'(x) dx.$$

The last equality tells you how you can treat limits of integration when you make a substitution. That is,

$$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du.$$

Note, you can either use substitution as before just to find an antiderivative and then go back to the old variable, or you can leave the substitution and change the limits of integration. Often I will ask you to use and show the change of the limits of integration.

Sample Problem: Change the limits of integration and evaluate $\int_0^1 \cos(\pi x^2)(2\pi x)dx$.
 Ans: Let $u = \pi x^2$, $du = 2\pi x dx$, when $x = 0$, $u = 0$, when $x = 1$, $u = \pi$, so

$$\int_0^1 \cos(\pi x^2)(2\pi x)dx = \int_0^\pi \cos u du = [\sin u]_0^\pi = 0.$$

Note that here we did not go back to the old variable x .

The point here is that each integral is an integral of a differential over the appropriate interval for that differential. If you think of the integral as a sum of differentials, then when you change the differential to an equivalent one, you change the limits of integration to correspond to the new interval.

EXAMPLES: To solve $\int_2^3 x^2 \sqrt{1+x^3} dx$ with the substitution $u = 1+x^3$, $\frac{1}{3} du = x^2 dx$, we note that when $x = 2$, $u = 9$, and when $x = 3$, $u = 28$. We then have

$$\int_2^3 x^2 \sqrt{1+x^3} dx = \frac{1}{3} \int_9^{28} \sqrt{u} du = \left[\frac{2}{9} u^{3/2} \right]_9^{28} = \frac{2}{9} (28^{3/2} - 27).$$

Here, the first integral is a limit of the “sum” of the differentials $x^2 \sqrt{1+x^3} dx$ as x goes from 2 to 3. With the substitution, $\frac{1}{3} \sqrt{u} du$, it becomes a limit of sums of equivalent differential with u going from 9 to 28.

Sometimes, the results of changing the limits of integration are surprising. For example, we can see that the total signed area between the graph of $y = x^3$ and the x -axis for $-5 \leq x \leq 5$ is 0. Also, we have $\int_{-5}^5 x^3 dx = \left[\frac{1}{4} x^4 \right]_{-5}^5 = 0$. To see this another way, we may look at $\int_{-5}^5 x^3 dx = \int_{-5}^5 x^2 \cdot x dx$ and set $u = x^2$, so that $\frac{1}{2} du = x dx$. Now when $x = -5$, $u = 25$, and when $x = 5$, $u = 25$, so

$$\int_{-5}^5 x^3 dx = \int_{-5}^5 x^2 \cdot x dx = \frac{1}{2} \int_{25}^{25} u du = 0$$

because the lower and upper limits of integration are the same.

EXAMPLE. Evaluate $\int_0^1 x e^{-x^2} dx$. Let $u = -x^2$. Then $-\frac{1}{2} du = x dx$, and the integral is

$$\int_0^{-1} \left(-\frac{1}{2}\right) e^u du = \frac{1}{2} \int_{-1}^0 e^u du = \frac{1}{2} [1 - e^{-1}].$$

EXAMPLE. Evaluate

$$\int_{-1}^1 (x^3 + x) \cdot \cos(2x^4 + 4x^2 + 1) dx.$$

Let $u = 2x^4 + 4x^2 + 1$. Then $\frac{1}{8} du = (x^3 + x) dx$, and the integral is $\frac{1}{8} \int_7^7 \cos u du = 0$. Since the original integrand is an odd function and the limits of integration are symmetric about 0, we could have predicted the answer.