1. A Tutoring Room is Open

7–9 p.m, Monday, Tuesday, Wednesday, Thursday, Room 140 Lincoln Hall.

2. Homework 20 due Thursday, November 2 at 9 a.m.

Section 5.5: #6, 10, 14, 22, 24, 28, 30, 34, 36, 44.

3. Homework 21 due Tuesday, November 7 at 9 a.m.

Section 5.6: #2, 6, 10, 14, 16, 22, 24, 26, 30, 36.

You may have to enter without evaluating. For example, write $(2^3)/3 - 1$ instead of

5/3.

4. Written problem for next week

Consider the region between the curves $y = x - 2$ and $y = 4 - x^2$. Set up and evaluate the integral or integrals for the area between the two curves in two ways, first integrate with respect to x, then integrate with respect to y. Note, this is a sample problem.

5. More on changing limits of integration

When you change variables in a definite integral, you change the limits of integration. Essentially, the integral is the limit of a sum of differentials. When you replace the differential with an equivalent one, the interval over which you are summing changes. For example, if $u = g(x)$ for x in $[a, b]$, then the interval for dx is $[a, b]$, and the interval for $du = g'(x)dx$ is the interval between $g(a)$ and $g(b)$ (we don't know in general which is bigger), so for a continuous function f on the latter interval,

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Often I will ask you to use and show the change of the limits of integration.

Sample Problem: Solve $\int_2^3 x^2\sqrt{1+x^3}\,dx$. **Ans.** Using the substitution $u = 1 + x^3$, $\frac{1}{3}du = x^2\,dx$, we note that when $x = 2$, $u = 9$, and when $x = 3$, $u = 28$. We then have

$$\int_2^3 x^2\sqrt{1+x^3}dx = \int_9^{28} \frac{1}{3}\sqrt{u}\,du = \left[\frac{2}{9}u^{\frac{3}{2}}\right]_9^{28} = \frac{2}{9}(28^{\frac{3}{2}} - 27).$$

Here, the first integral is a limit of the "sum" of the differentials $x^2\sqrt{1+x^3}dx$ as x goes from 2 to 3. With the substitution, $\frac{1}{3}\sqrt{u}\,du$, it becomes a limit of the sum of an equivalent differential with u going from 9 to 28.

Sometimes, the results of changing the limits of integration are surprising. For example, we can see that the total signed area between the graph of $y = x^3$ and the x-axis for $-5 \leq x \leq 5$ is 0. Also, we have $\int_{-5}^{5} x^3 dx = \left[\frac{1}{4}x^4\right]_{-5}^{5} = 0$. To see this another way, we may look at $\int_{-5}^{5} x^3 dx = \int_{-5}^{5} x^2 \cdot x\,dx$ and set $u = x^2$, so that $\frac{1}{2}du = x\,dx$. Now when $x = -5$, $u = 25$, and when $x = 5$, $u = 25$, so

$$\int_{-5}^{5} x^3 dx = \int_{-5}^{5} x^2 \cdot x\,dx = \frac{1}{2}\int_{25}^{25} u\,du = 0$$

because the lower and upper limits of integration are the same.

EXAMPLE. Evaluate $\int_0^1 xe^{-x^2}dx$. Let $u = -x^2$. Then $-\frac{1}{2}du = xdx$, and the integral is

$$\int_0^{-1} (-\frac{1}{2})e^u du = \frac{1}{2}\int_{-1}^{0} e^u du = \frac{1}{2}\left[1 - e^{-1}\right].$$

Note here that after the substitution, the limits of integration are in reverse order.

EXAMPLE. Evaluate

$$\int_{-1}^{1} (x^3 + x) \cdot \cos(2x^4 + 4x^2 + 1)\ dx.$$

Let $u = 2x^4 + 4x^2 + 1$. Then $\frac{1}{8}du = (x^3 + x)dx$, and the integral is $\frac{1}{8}\int_7^7 \cos u\ du = 0$. Since the original integrand is an odd function and the limits of integration are symmetric about 0, we could have predicted the answer as shown next.

6. INTEGRALS OF EVEN AND ODD FUNCTIONS.

Definition 1. *A function $f(x)$ is even if for each x for which f is defined, f is defined at $-x$ and $f(-x) = f(x)$. A function $f(x)$ is odd if for each x for which f is defined, f is defined at $-x$ and $f(-x) = -f(x)$.*

Example: $\cos x$ is even and $\sin x$ is odd.

If $f(x)$ is an odd function on an interval $[-a, a]$, we have $\int_{-a}^{a} f(x)dx = 0$. This is seen either graphically or using substitution. For the latter proof, note that for $u = -x$, $du = -dx$, and so

$$\int_{x=-a}^{x=0} f(x)dx = \int_{u=a}^{u=0} -f(-u)du = \int_{u=a}^{u=0} f(u)du = -\int_{u=0}^{u=a} f(u)du.$$

Therefore, just as can be seen graphically, the integrals over $[-a, 0]$ and $[0, a]$ have opposite signs.

If $f(x)$ is an even function on an interval $[-a, a]$, we have $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$. This is seen either graphically or using substitution. For the latter proof, note that for $u = -x$, $du = -dx$, and so

$$\int_{x=-a}^{x=0} f(x)dx = \int_{u=a}^{u=0} -f(-u)du = \int_{u=a}^{u=0} -f(u)du = \int_{u=0}^{u=a} f(u)du.$$

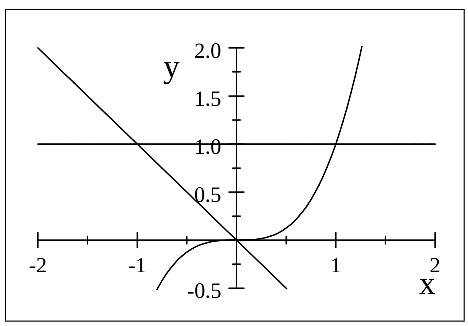
Therefore, just as can be seen graphically, the integrals over $[-a, 0]$ and $[0, a]$ have the same value.

7. Area Between Curves

You will have homework on the area between curves. Some problems of this type we can work either as integrals with respect to x or as integrals with respect to y, or as both.

EXAMPLE: Find the region bounded by the curves $y = x^3$, $y = -x$, and $y = 1$ and find the area of the region.

x^3



As shown, the curves intersect in pairs at $(-1, 1)$, $(0, 0)$, and $(1, 1)$. If we integrate with respect to x, we need two integrals for the area:

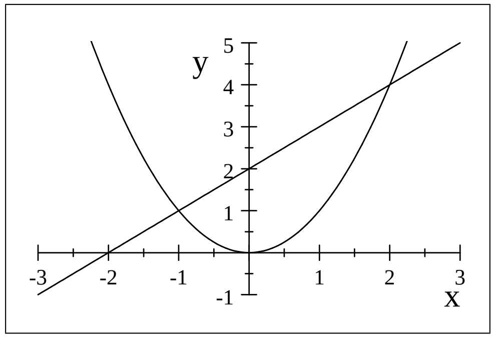
$$\begin{aligned}\int_{x=-1}^{x=0} (1-(-x))\,dx + \int_{x=0}^{x=1} \left(1-x^3\right)dx &= \left[x+\frac{1}{2}x^2\right]_{-1}^{0} + \left[x-\frac{1}{4}x^4\right]_0^1 \\ &= 0-\left(-1+\frac{1}{2}\right)+1-\frac{1}{4} = \frac{5}{4}.\end{aligned}$$

If we integrate with respect to y, we only need one integral for the area. Here it is the function for the right most graph minus the other function:

$$\int_{y=0}^{y=1} \left(y^{1/3}-(-y)\right)dy = \left[\frac{3}{4}y^{4/3}+\frac{1}{2}y^2\right]_0^1 = \frac{3+2}{4} = \frac{5}{4}.$$

Example: Consider the region between the curves $y = x^2$ and $y = x + 2$.

x^2



a) Plot these two curves and show the region between the curves.

b) Set up and evaluate the integral or integrals for the area between the two curves, integrating with respect to x. The two curves intersect when $x^2 - x - 2 = 0$, i.e., $x = -1$ and $x = 2$. The area is

$$\int_{-1}^{2} \left(x + 2 - x^2\right) dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2} = 2 + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3} = 4.5 = \frac{9}{2}.$$

c) Set up and evaluate the integral or integrals for the area between the two curves, integrating with respect to y.

$$\begin{aligned} \int_0^1 2\sqrt{y}dy + \int_1^4 \left(\sqrt{y} - (y-2)\right) dy &= \left[\frac{4}{3}y^{\frac{3}{2}}\right]_0^1 + \left[\frac{2}{3}y^{\frac{3}{2}} - \frac{1}{2}y^2 + 2y\right]_1^4 \\ &= \frac{4}{3} + \frac{16}{3} - \frac{16}{2} + 8 - \frac{2}{3} + \frac{1}{2} - 2 = \frac{9}{2}. \end{aligned}$$

8. Justification for the integral for the area between two curves

We can justify these area calculations as follows.

Suppose f and g are continuous functions on $[a, b]$ with $f(x) \leq g(x)$ for all x in $[a, b]$. We want to show that the area A between the two graphs on $[a, b]$ is the integral $\int_a^b (g(x) - f(x))\, dx$. Given $\Delta x > 0$, let M_i^f, m_i^f, M_i^g, m_i^g be the maxima and minima of f and g on $[x_{i-1}, x_i]$. Let A_i be the area between the two curves on $[x_{i-1}, x_i]$. Then

$$\begin{aligned} \left(m_i^g - M_i^f\right) \cdot \Delta x_i &\leq A_i \leq \left(M_i^g - m_i^f\right) \cdot \Delta x_i, \\ \left(m_i^g - M_i^f\right) \cdot \Delta x_i &\leq (g(x_{i-1}) - f(x_{i-1})) \cdot \Delta x_i \leq \left(M_i^g - m_i^f\right) \cdot \Delta x_i. \end{aligned}$$

Therefore,

$$\begin{aligned} |A_i - (g(x_{i-1}) - f(x_{i-1}))| \cdot \Delta x_i &\leq \Big((M_i^g - m_i^g) + \Big(M_i^f - m_i^f\Big)\Big) \cdot \Delta x_i \\ &\leq (E_g(\Delta x) + E_f(\Delta x)) \cdot \Delta x_i. \end{aligned}$$

Since $E_g(\Delta x) + E_f(\Delta x)$ has limit 0 as $\Delta x \to 0$, the integral is the correct calculation by the error sum principle.