1. A Tutoring Room is Open

7–9 p.m, Monday, Tuesday, Wednesday, Thursday, Room 140 Lincoln Hall.

2. Homework 21 due Tuesday, November 7 at 9 a.m.

Section 5.6: #2, 6, 10, 14, 16, 22, 24, 26, 30, 36.

You may have to enter without evaluating. For example, write $(2^3)/3 - 1$ instead of

5/3.

3. Homework 22 due Thursday, November 9 at 9 a.m.

Section 5.7: #4, 10, 14, 38, 40, 44, 48, 52, 54, 56, 58, 60.

You may have to enter without evaluating. For example, write $(2^3)/3 - 1$ instead of 5/3.

4. Written problem for next week

Consider the region between the curves $y = x - 2$ and $y = 4 - x^2$. Set up and evaluate the integral or integrals for the area between the two curves in two ways, first integrate with respect to x, then integrate with respect to y. Note, this is a sample problem.

5. Another Substitution Problem

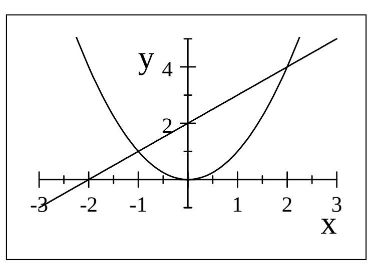
Sample Problem: For the integral $\int x(x+3)^7 dx$, we would like to simplify $x + 3$. Therefore, we let $u = x + 3$, so $x = u - 3$, and $du = dx$. We then have

$$\int x(x+3)^7 dx = \int (u-3)u^7 du = \int u^8 - 3u^7 du.$$

6. Example on Area Between Curves

Sample Problem. Consider the region between the curves $y = x^2$ and $y = x + 2$. Set up and evaluate the integral or integrals for the area between the two curves in two ways, first integrate with respect to x, then integrate with respect to y.

x^2



Ans:

a) First plot the two curves and show the region between the curves.

b) Set up and evaluate the integral or integrals for the area between the two curves, integrating with respect to x. The two curves intersect when $x^2 - x - 2 = 0$, i.e., $x = -1$ and $x = 2$. The area is

$$\int_{-1}^{2} \left(x + 2 - x^2\right) dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2} = 2 + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3} = 4.5 = \frac{9}{2}.$$

c) Set up and evaluate the integral or integrals for the area between the two curves, integrating with respect to y.

$$\begin{aligned}\int_0^1 2\sqrt{y}dy + \int_1^4 \left(\sqrt{y} - (y-2)\right) dy &= \left[\frac{4}{3}y^{\frac{3}{2}}\right]_0^1 + \left[\frac{2}{3}y^{\frac{3}{2}} - \frac{1}{2}y^2 + 2y\right]_1^4 \\ &= \frac{4}{3} + \frac{16}{3} - \frac{16}{2} + 8 - \frac{2}{3} + \frac{1}{2} - 2 = \frac{9}{2}.\end{aligned}$$

7. TRAPEZOIDAL APPROXIMATION TO INTEGRALS

If one wants to obtain a numerical approximation for the value of an integral, there are better sums than Riemann sums. The next simplest sum for approximating $\int_a^b f(x)dx$ is given by the Trapezoidal rule. For this sum, Δx exactly divides the interval $[a, b]$, that is, Δx is equal to $(b - a)/n$ for some integer n. In each interval $[x_{i-1}, x_i]$ one averages the values of the function at the two endpoints of the interval. This gives the area of the trapezoid with base on the interval $[x_{i-1}, x_i]$, left height $f(x_{i-1})$, right height $f(x_i)$. Except for a and b, each value $f(x_j)$ is multiplied by $1/2$ but added twice, so the trapezoidal sum is the same as the Riemann sum except the value of f at a is multiplied by $1/2$, and the value of f at b is included, but multiplied by $1/2$. Your book multiplies the value at every point except a and b by 2 and then divides the total by 2. This is a stupid thing to do for a numerical approximation because it is more work.

Sample Problem: Find the value of the integral $\int_1^2 x^2 dx$, and for $\Delta x = \frac{1}{5}$ write out the Riemann sum (evaluating at the left) and the trapezoidal sum using the method discussed in class. Do not simplify.

The value of the integral is $\int_1^2 x^2 dx = \frac{8-1}{3} = \frac{7}{3} = 2.33333$.

The Riemann sum with $\Delta x = \frac{1}{5}$ is

$$\left(1^2 + (1.2)^2 + (1.4)^2 + (1.6)^2 + (1.8)^2\right)/5.$$

(This is 2.04.)

The trapezoidal approximation is

$$\left(1^2 \cdot \frac{1}{2} + (1.2)^2 + (1.4)^2 + (1.6)^2 + (1.8)^2 + 2^2 \cdot \frac{1}{2}\right)/5.$$

(This is 2.34.)

Note that the book would have you do the following calculation:

$$\left(1^2 + 2 \cdot (1.2)^2 + 2 \cdot (1.4)^2 + 2 \cdot (1.6)^2 + 2 \cdot (1.8)^2 + 2^2\right)/2 \cdot 5.$$

This gives the same value, but involves more calculations. If you used $\Delta x = 1/100$, you would have to do 99 multiplications by 2 instead of 2 multiplications by 1/2.

8. Application to finding the mass of a rod

Suppose we have a rod lying on the x-axis between a and b. A continuous function $\rho(x)$ is called a **mass density** for the rod when the following is true: For any subinterval $[c, d]$ of $[a, b]$, the mass of the rod between c and d is given by the integral $\int_c^d \rho(x)dx$. As we have seen, this means that for any x in $[a, b)$ and any small $\Delta x > 0$, the mass of the rod between x and $x + \Delta x$ is $\int_x^{x+\Delta x} \rho(t)dt = \rho(c) \cdot \Delta x$ for some c between x and $x + \Delta x$. Therefore, the average of the mass from x to $x + \Delta x$ converges to $\rho(x)$ as $\Delta x \to 0$.

We may assume, but it is not necessary, that the left end of the rod is at $x = 0$ and the right end is at L. Let S be the **total** mass of the rod. It is clear that any upper sum for the density is greater than S, and any lower sum for the density is smaller than S. It follows that the integral of the density is S. That is, $S = \int_0^L \rho(x)\ dx$. A similar calculation works for a weight density to give the weight of the rod.

Sample Problem. Suppose the mass density of a 10 centimeter rod with the left end at 0 is given by $\rho(x) = x^2$ grams/cm. Find the mass of the rod. Ans. The mass is $\int_0^{10} x^2 dx = \frac{1000}{3}$ grams.

Example. Suppose a rod in the previous problem has length 2 centimeters and the mass density in grams per centimeter is given by

$$\rho(x) = \sqrt{3x^2 + 6x} \cdot (x + 1).$$

Then the total mass of the rod is

$$\int_0^2 \rho(x)\ dx = \int_0^2 \sqrt{3x^2 + 6x} \cdot (x + 1)\ dx.$$

Letting $u = 3x^2 + 6x$, we have $\frac{1}{6}du = (x + 1)dx$, so the total mass of the rod is

$$\frac{1}{6}\int_0^{24} \sqrt{u}\ du = \frac{1}{9}\left[u^{3/2}\right]_0^{24} = \frac{24^{3/2}}{9} = 13.0639 \text{ grams}.$$

Note the change of the limits of integration.

9. Net and Total Distance

Velocity is the time derivative of position. Over a small interval of time $[t_{i-1}, t_i]$ the signed change in position $x(t_i) - x(t_{i-1}) = x'(c_i)\Delta t_i = v(c_i)\Delta t_i$ for some time c_i in the interval. It follows that the **Net** Distance traveled from time 0 to time T is

$$\lim_{\Delta t \to 0} \sum_{i=1}^{n} (x(t_i) - x(t_{i-1})) = \lim_{\Delta t \to 0} \sum_{i=1}^{n} v(c_i)\Delta t_i = \int_0^T v(t)\ dt.$$

For the total distance traveled, one has to integrate speed, which is the absolute value of the velocity.

Sample Problem. Suppose the velocity of a moving body at time t is given by $v(t) = 1 - t$ for $0 \leq t \leq 2$. Find the net distance traveled and the total distance traveled for that interval of time.

Ans. The net distance traveled is

$$\int_0^2 1 - t\ dt = \left[t - \frac{1}{2}t^2 \right]_0^2 = 0,$$

but the total distance traveled is

$$\begin{aligned} \int_0^2 |1 - t|\ dt &= \int_0^1 1 - t\ dt + \int_1^2 t - 1\ dt \\ &= \left[t - \frac{1}{2}t^2 \right]_0^1 + \left[\frac{1}{2}t^2 - t \right]_1^2 \\ &= \frac{1}{2} + \frac{1}{2} = 1. \end{aligned}$$