1. A TUTORING ROOM IS OPEN

7–9 p.m, Monday, Tuesday, Wednesday, Thursday, Room 140 Lincoln Hall.

2. HOMEWORK 22 DUE THURSDAY, NOVEMBER 9 AT 9 A.M.

Section 5.7: #4, 10, 14, 38, 40, 44, 48, 52, 54, 56, 58, 60.

You may have to enter without evaluating. For example, write $(2^3)/3 - 1$ instead of 5/3.

3. HOMEWORK 23 DUE TUESDAY, NOVEMBER 14 AT 9 A.M.

Section 5.8: #2, 4, 8, 14, 18, 20, 24, 26, 28, 36.

4. HOMEWORK 24 DUE THURSDAY, NOVEMBER 16 AT 9 A.M.

Section 5.9: #2, 4, 6. The notation T_n denotes the trapezoidal approximation to the integral. The subscript n means the the interval $[a, b]$ is divided into n intervals, so $\Delta x = (b - a)/n$.

Section 6.1: #16, 18, 20, 30, 38.

Section 6.2: #2, 4, 6.

5. WRITTEN PROBLEM FOR NEXT WEEK

Let $R(b)$ be the region between the x-axis and the curve $y = 1/x$ for $1 \leq x \leq b$.

a) What is the area $A(b)$ of the region $R(b)$?

b) If you rotate the region $R(b)$ about the x-axis, what is the resulting volume $V(b)$?

c) What is the limit of the area $A(b)$ as $b \to +\infty$?

d) What is the limit of the volume $V(b)$ as $b \to +\infty$?

6. SOLIDS OF REVOLUTION, METHOD OF WASHERS

We often rotate the area between two curves about a line. If the line is the x-axis, and $0 < g(x) < f(x)$ between a and b then the cross section is a disk of radius $f(x)$ from which has been removed a disk of radius $g(x)$. The cross section has area $A(x) = \pi[(f(x))^2 - (g(x))^2]$. **NOTE: It is not the difference squared times π!** The volume of the solid is

$$V = \int_a^b \pi[(f(x))^2 - (g(x))^2]\, dx.$$

If instead of the x-axis, we rotate the area between the two curves about a line parallel to the x-axis and a distance d below the x-axis, then we replace $f(x)$ with $f(x) + d$, and we replace $g(x)$ with $g(x) + d$. Instead of adding d, we subtract d if the line is above the x-axis but below the area we want to rotate. We work with similar calculations if we want to rotate about the y-axis or a line parallel to the y-axis.

Sample Problem. Find the volumes of the two solids you get by rotating the area between the line $y = x$ and $y = x^2$ about the x-axis and about the y-axis. **Ans.** The volume of the first solid is

$$\pi \int_0^1 [x^2 - x^4]\,dx = \pi \left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1 = \pi\frac{2}{15}.$$

For the second solid, we want to rotate the area between the line $x = y$ and the curve $x = \sqrt{y}$ about the y-axis. The volume of the corresponding solid is

$$\pi \int_0^1 [y - y^2]\,dy = \pi \left[\frac{1}{2}y^2 - \frac{1}{3}y^3\right]_0^1 = \pi\frac{1}{6}.$$

Example: We want to rotate the area between the line $y = x$ and $y = x^2$ about the line $y = -2$. The volume of the corresponding solid is

$$\begin{aligned} \pi \int_0^1 [(x+2)^2 - (x^2+2)^2]\,dx &= \pi \int_0^1 -x^4 - 3x^2 + 4x\,dx \\ &= \pi(-\frac{1}{5} - 1 + 2) = \frac{4\pi}{5}. \end{aligned}$$

7. WORK

In this section I will stay with the English system of units. Metric units are discussed in your book. The unit of force in the English system is the pound. When we say something weighs one pound, we are speaking of the force exerted on the object by the gravitational field of the Earth. If a constant force acts on an object in a fixed direction and moves the object over a certain distance, we define the **work** done as the product of the magnitude of the force times the distance the object is moved. For example, if I lift a 5 pound object 3 feet off of the ground, I have done 15 foot-pounds of work. If I push a heavy box along a rough floor for 7 feet and I must apply a constant force of 11 pounds to do so, then I have done 77 foot-pounds of work. Here, I am assuming that none of the force is used to lift up the box and all of it is in the direction of motion.

What do we do if the force changes over the duration of the motion? Assume that $F(x)$ is the force applied at x over an interval $[a, b]$. Fix a positive Δx and let $[x_{i-1}, x_i]$ be the i^{th} interval of the Δx partition. If W_i is the work done in moving the object from x_{i-1} to x_i, and $\overline{F}_i$ is the maximum force in the interval while $\underline{F}_i$ is the minimum force, then $\underline{F}_i \cdot \Delta x_i \le W_i \le \overline{F}_i \cdot \Delta x_i$, and so it follows that the total work done is

$$W = \int_a^b F(x)\,dx.$$

Sample Problem. Find the total work done in applying a force $F(x) = 3x$ pounds while compressing a spring from its relaxed position at $x = 0$ to the position $x = 6$ inches. **Ans.** The total work done is

$$W = \int_0^{1/2} 3x\,dx = \frac{3}{2}x^2\Big]_0^{1/2} = \frac{3}{8} \text{ foot-pounds.}$$

If, as in a spring, the force you exert changes over the distance you move, then you calculate the work done for each small interval. For example, suppose you want to lift a leaky bucket of water 10 feet off of the ground. At height y, the bucket weighs $10 - y$ pounds. What is the work you do? In lifting the bucket from y to $y + \Delta y$, the work you do is between $(10 - y - \Delta y) \cdot \Delta y$ and $(10 - y) \cdot \Delta y$. Choosing $(10 - y) \cdot \Delta y$ for the work done for that interval, we have an error that is at most $\Delta y \cdot \Delta y$, and the sum of these errors goes to 0 as Δy goes to 0. In differential notation, the work done is $(10 - y)dy$ plus a small error with the sum of the errors going to 0. The total work is

$$\int_0^{10} (10 - y)dy = \left[10y - \frac{1}{2}y^2\right]_0^{10} = 50 \text{ foot-pounds.}$$

An application of work involves lifting fluid from a tank to a given height H. Here, y is the natural variable with which to set up the problem. We assume that the fluid to be moved occupies a solid space associated with an interval $[a, b]$ on the y-axis with $H \geq b$. We will assume that the weight density ρ of the fluid is constant; this represents the pounds per unit volume of the fluid. For each y, let $A(y)$ be the area of a cross section of the fluid. At each y, the distance you must move the fluid is $H - y$. It follows that for an increment dy of the interval $[a, b]$, the increment of volume $dV = A(y)dy$, the increment of weight $dw = \rho \cdot A(y)\,dy$, and the increment of work done is $dW = \rho \cdot (H - y) \cdot A(y)\,dy$. Therefore, the total work done is

$$W = \int_a^b \rho \cdot (H - y) \cdot A(y)\,dy.$$

If we are filling a tank by lifting a liquid from $y = 0$, then the integral we want is

$$W = \int_a^b \rho \cdot y \cdot A(y)\,dy.$$

For water, we use the value $\rho = 62.4$ pounds/ft^3.

EXAMPLE: If we want to fill a cylindrical tank of radius r and height h by lifting water from the level $y = 0$, then for a given level of y between 0 and h, the cross section is πr^2, an increment of volume is $dV = \pi r^2 dy$, the weight is $dw = 62.4dV = 62.4 \cdot \pi r^2 dy$, and the increment of work is

$$dW = y \cdot dw = 62.4ydV = 62.4 \cdot \pi r^2 y\,dy.$$

The total work done is

$$\int_0^h dW = \int_0^h 62.4 \cdot \pi r^2 y\, dy = 62.4 \cdot \pi r^2 \frac{h^2}{2}.$$

Sample Problem. Write the integral for the total work necessary to empty a half spherical tank of water of radius r resting on the ground, where the water must be moved to the point $y = H \geq r$. First, we note that each cross section below the level $y = r$ has radius

$$s = \sqrt{r^2 - (r-y)^2} = \sqrt{2yr - y^2}.$$

The total work done is given by the integral

$$62.4\pi \int_0^r (H - y)(2yr - y^2)\, dy.$$

Note: If a tank is being emptied from the bottom by gravity, then it is gravity that is doing the work.

Often, there is more than one way to set up a problem.

Problem: A rope is 80 feet long and weighs 3/10 of a pound per foot. It has one end at the top of a tall building and the other end hanging down 80 feet from the top of the building. The rope is to be pulled onto the top of the building. Calculate the work.

Ans: If we let y be the distance to the bottom end of the rope, then the section from y to $y + dy$ weighs 3/10 dy of a pound and must be moved $80 - y$ feet, so the work is

$$\int_0^{80} \frac{3}{10}(80 - y)dy = 960 \text{ foot-pounds}.$$

If we let x be the length of rope that has been pulled over the side of the building, then in lifting from x to $x + dx$, the weight is $\frac{3}{10}(80 - x)$, and it is moved dx, so we get the same answer.