

## 1. A TUTORING ROOM IS OPEN

7–9 p.m, Monday, Tuesday, Wednesday, Thursday, Room 140 Lincoln Hall.

## 2. HOMEWORK 23 DUE TUESDAY, NOVEMBER 14 AT 9 A.M.

Section 5.8: #2, 4, 8, 14, 18, 20, 24, 26, 28, 36.

## 3. HOMEWORK 24 DUE THURSDAY, NOVEMBER 16 AT 9 A.M.

Section 5.9: #2, 4, 6. The notation  $T_n$  denotes the trapezoidal approximation to the integral. The subscript  $n$  means the the interval  $[a, b]$  is divided into  $n$  intervals, so  $\Delta x = (b - a)/n$ .

Section 6.1: #16, 18, 20, 30, 38.

Section 6.2: #2, 4, 6.

## 4. WRITTEN PROBLEM FOR NEXT WEEK

Let  $R(b)$  be the region between the  $x$ -axis and the curve  $y = 1/x$  for  $1 \leq x \leq b$ .

- a) What is the area  $A(b)$  of the region  $R(b)$ ?
- b) If you rotate the region  $R(b)$  about the  $x$ -axis, what is the resulting volume  $V(b)$ ?
- c) What is the limit of the area  $A(b)$  as  $b \rightarrow +\infty$ ?
- d) What is the limit of the volume  $V(b)$  as  $b \rightarrow +\infty$ ?

## 5. WORK

In this section I will stay with the English system of units. Metric units are discussed in your book. The unit of force in the English system is the pound. When we say something weighs one pound, we are speaking of the force exerted on the object by the gravitational field of the Earth. If a constant force acts on an object in a fixed direction and moves the object over a certain distance, we define the **work** done as the product of the magnitude of the force times the distance the object is moved. For example, if I lift a 5 pound object 3 feet off of the ground, I have done 15 foot-pounds of work. If I push a heavy box along a rough floor for 7 feet and I must apply a constant force of 11 pounds to do so, then I have done 77 foot-pounds of work. Here, I am assuming that none of the force is used to lift up the box and all of it is in the direction of motion.

What do we do if the force changes over the duration of the motion? Assume that  $F(x)$  is the force applied at  $x$  over an interval  $[a, b]$ . Fix a positive  $\Delta x$  and let  $[x_{i-1}, x_i]$  be the  $i^{\text{th}}$  interval of the  $\Delta x$  partition. If  $W_i$  is the work done in moving the object from  $x_{i-1}$  to  $x_i$ , and  $\overline{F}_i$  is the maximum force in the interval while  $\underline{F}_i$  is the minimum force, then  $\underline{F}_i \cdot \Delta x_i \leq W_i \leq \overline{F}_i \cdot \Delta x_i$ , and so it follows that the total work done is

$$W = \int_a^b F(x) dx.$$

**Sample Problem.** Find the total work done in applying a force  $F(x) = 3x$  pounds while compressing a spring from its relaxed position at  $x = 0$  to the position  $x = 6$  inches. **Ans.** The total work done is

$$W = \int_0^{1/2} 3x \, dx = \left. \frac{3}{2}x^2 \right|_0^{1/2} = \frac{3}{8} \text{ foot-pounds.}$$

If, as in a spring, the force you exert changes over the distance you move, then you calculate the work done for each small interval. For example, suppose you want to lift a leaky bucket of water 10 feet off of the ground. At height  $y$ , the bucket weighs  $10 - y$  pounds. What is the work you do? In lifting the bucket from  $y$  to  $y + \Delta y$ , the work you do is between  $(10 - y - \Delta y) \cdot \Delta y$  and  $(10 - y) \cdot \Delta y$ . Choosing  $(10 - y) \cdot \Delta y$  for the work done for that interval, we have an error that is at most  $\Delta y \cdot \Delta y$ , and the sum of these errors goes to 0 as  $\Delta y$  goes to 0. In differential notation, the work done is  $(10 - y)dy$  plus a small error with the sum of the errors going to 0. The total work is

$$\int_0^{10} (10 - y)dy = \left[ 10y - \frac{1}{2}y^2 \right]_0^{10} = 50 \text{ foot-pounds.}$$

An application of work involves lifting fluid from a tank to a given height  $H$ . Here,  $y$  is the natural variable with which to set up the problem. We assume that the fluid to be moved occupies a solid space associated with an interval  $[a, b]$  on the  $y$ -axis with  $H \geq b$ . We will assume that the weight density  $\rho$  of the fluid is constant; this represents the pounds per unit volume of the fluid. For each  $y$ , let  $A(y)$  be the area of a cross section of the fluid. At each  $y$ , the distance you must move the fluid is  $H - y$ . It follows that for an increment  $dy$  of the interval  $[a, b]$ , the increment of volume  $dV = A(y)dy$ , the increment of weight  $dw = \rho \cdot A(y)dy$ , and the increment of work done is  $dW = \rho \cdot (H - y) \cdot A(y)dy$ . Therefore, the total work done is

$$W = \int_a^b \rho \cdot (H - y) \cdot A(y) \, dy.$$

If we are filling a tank by lifting a liquid from  $y = 0$ , then the integral we want is

$$W = \int_a^b \rho \cdot y \cdot A(y) \, dy.$$

For water, we use the value  $\rho = 62.4$  pounds/ft<sup>3</sup>.

**EXAMPLE:** If we want to fill a cylindrical tank of radius  $r$  and height  $h$  by lifting water from the level  $y = 0$ , then for a given level of  $y$  between 0 and  $h$ , the cross section is  $\pi r^2$ , an increment of volume is  $dV = \pi r^2 dy$ , the weight is  $dw = 62.4dV = 62.4 \cdot \pi r^2 dy$ , and the increment of work is

$$dW = y \cdot dw = 62.4y dV = 62.4 \cdot \pi r^2 y \, dy.$$

The total work done is

$$\int_0^h dW = \int_0^h 62.4 \cdot \pi r^2 y \, dy = 62.4 \cdot \pi r^2 \frac{h^2}{2}.$$

**Sample Problem.** Write the integral for the total work necessary to empty a half spherical tank of water of radius  $r$  resting on the ground, where the water must be moved to the point  $y = H \geq r$ . First, we note that each cross section below the level  $y = r$  has radius

$$s = \sqrt{r^2 - (r - y)^2} = \sqrt{2yr - y^2}.$$

The total work done is given by the integral

$$62.4\pi \int_0^r (H - y)(2yr - y^2) \, dy.$$

**Note:** If a tank is being emptied from the bottom by gravity, then it is gravity that is doing the work.

Often, there is more than one way to set up a problem.

**Problem:** A rope is 80 feet long and weighs  $3/10$  of a pound per foot. It has one end at the top of a tall building and the other end hanging down 80 feet from the top of the building. The rope is to be pulled onto the top of the building. Calculate the work.

**Ans:** If we let  $y$  be the distance to the bottom end of the rope, then the section from  $y$  to  $y + dy$  weighs  $3/10 \, dy$  of a pound and must be moved  $80 - y$  feet, so the work is

$$\int_0^{80} \frac{3}{10} (80 - y) \, dy = 960 \text{ foot-pounds.}$$

If we let  $x$  be the length of rope that has been pulled over the side of the building, then in lifting from  $x$  to  $x + dx$ , the weight is  $\frac{3}{10}(80 - x)$ , and it is moved  $dx$ , so we get the same answer.

## 6. CENTROIDS

In balancing a teeter-totter, about a fulcrum, you know that both weight and distance from the fulcrum must be considered. If you let  $w$  be the sum of the weights,  $w = \sum w_i$ , when the sum  $\sum \frac{w_i}{w} = 1$ . Setting the origin where you like, you then consider the signed sum  $\bar{x} = \sum \frac{w_i}{w} x_i$ . This is an “average” of the  $x$ -values, and to get a balance, the fulcrum must be placed at  $\bar{x}$ .

If we think of these weights as being on the  $x$ -axis in the plane, then we set  $M_y = \sum w_i x_i$  and call this the moment about the  $y$ -axis, and  $\bar{x} = M_y/w$ .

Now think of a thin plate or “lamina” between two curves  $y = f(x)$  and  $y = g(x)$  from  $a$  to  $b$ , with a constant weight density  $\rho$  in terms of weight per unit area. (The second curve might be the  $x$ -axis, i.e.,  $y = 0$ .) Then for any  $\Delta x > 0$  and corresponding piece of the plate from  $x_{i-1}$  to  $x_i$ , the weight of the piece is  $\rho A_i$  where  $A_i$  is the area of the piece. The total weight is  $\sum \rho A_i$ , so the ratio  $\rho A_i / \sum \rho A_i$  is just  $A_i / A$ , where  $A$  is the total area of the piece  $\int_a^b (f(x) - g(x)) dx$ . Now the “average”  $x$ -value

$$\bar{x} = \int_a^b x \frac{dA}{A} = \frac{1}{A} \int_a^b x \cdot (f(x) - g(x)) dx.$$

The value  $\bar{x}$  is the  $x$ -coordinate of the centroid of the plate. The (area) **moment** about the  $y$ -axis is  $M_y = \int_a^b x dA = \int_a^b x \cdot (f(x) - g(x)) dx$ .

The centroid of the strip from  $x_{i-1}$  to  $x_i$  is half way from  $g(x)$  to  $f(x)$ . That is,  $\bar{y}_i = \frac{1}{2}(f(x) + g(x))$ . Note that  $\frac{1}{2}(f(x) + g(x)) = g(x) + \frac{1}{2}(f(x) - g(x))$ . This means that the (area) moment of the strip about the  $x$ -axis is

$$\bar{y}_i dA = \frac{1}{2}(f(x) + g(x)) dA = \frac{1}{2}(f(x) + g(x))(f(x) - g(x)) dx = \frac{1}{2}((f(x))^2 - (g(x))^2) dx.$$

These add, so the (area) moment about the  $x$ -axis is

$$M_x = \frac{1}{2} \int_a^b ((f(x))^2 - (g(x))^2) dx.$$

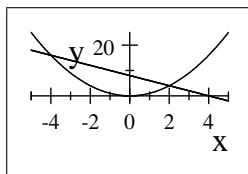
The average  $y$  value  $\bar{y}$  is  $M_x / A$ .

Note that instead of working with area, we get the same calculations if we use weight, but take  $\rho \equiv 1$ . Any other constant value of  $\rho$  will appear in both the numerator and denominator for the ratio calculating  $\bar{x}$  and  $\bar{y}$ , so we get the same result as just taking area.

A flat piece of metal will balance on a pin head if the pin head is at the centroid.

**Sample Problem:** For the region between the curves  $y = 8 - 2x$  and the curve  $y = x^2$  from 0 to 2, find the area, the moments about the  $y$  and  $x$  axes, and the centroid.

$x^2$



**Ans:** the area is

$$A = \int_0^2 dA = \int_0^2 (8 - 2x - x^2) dx = \frac{28}{3}.$$

The moment about the  $y$ -axis is

$$M_y = \int_0^2 x dA = \int_0^2 (x \cdot (8 - 2x - x^2)) dx = \frac{20}{3}.$$

The moment about the  $x$ -axis is

$$M_x = \frac{1}{2} \int_0^2 ((8 - 2x)^2 - (x^2)^2) dx = \frac{512}{15}.$$

The centroid is given by

$$\begin{aligned}\bar{x} &= \frac{M_y}{A} = \frac{\frac{20}{3}}{\frac{28}{3}} = \frac{5}{7} = 0.7143 \\ \bar{y} &= \frac{M_x}{A} = \frac{\frac{512}{15}}{\frac{28}{3}} = \frac{128}{35} = 3.6571.\end{aligned}$$