1. A Tutoring Room is Open

7–9 p.m, Monday, Tuesday, Wednesday, Thursday, Room 140 Lincoln Hall.

2. Homework 26 due Thursday, November 30 at 9 a.m.

Section 6.5: #18, 20, 26, 28.

Section 6.6: #6, 8, 10, 12.

3. Homework 27 due Tuesday, December 5 at 9 a.m.

Section 6.7: #4, 6, 12, 18, 26, 28.

Section 6.8: #10, 22, 32, 36, 38, 42.

4. Homework 28 due Thursday, December 7 at 9 a.m.

Section 8.1: #2, 6, 22, 24, 26, 28, 30, 38.

5. Written problem for next week

Here is a problem using the arctan function. A rectangular painting is hung on a wall. How high should the painting be so that the top and bottom subtend the maximum angle for the viewer. We may assume that the viewer is a feet from the picture, and the picture is b feet high. We will also assume that the viewer's eyes are at level $y = 0$, and that the bottom of the painting is at y, which may be positive or negative or 0.

6. Final Exam, Monday December 11, 8-11 a.m.

Section 4 (Liu Qi), Section 5 (Liu Qi) Section 6 (Michael Barrus), Section 8 (Scott Weaver) will take the exam in Room 314 Altgeld Hall.

Section 2 (Isaac Goldbring) and half of Section 7 (Isaac Goldbring) (Names beginning with A-G) will take the exam in Room 32 of the Psychology Building. The other half of Section 7 (Isaac Goldbring) and Section 9 (Timothy LeSaulnier) will take the exam in Room 142 of the Psychology Building. People in these sections **must** go to this room and not Altgeld Hall to take the exam.

Everyone should by now know their discussion section and section instructor. You will need to enter that on your examination. Bring your U of I identity card to show when turning in the exam.

7. Arctan Review

The $\arctan x = \tan^{-1} x$ is the angle whose tangent is x. The domain is the whole real line, but the values taken are between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. To find the derivative, we set $y = \arctan x$, so $\tan y = x$. Differentiating both sides with respect to x (and using the chain rule), we get $\sec^2 y \cdot \frac{dy}{dx} = 1$. That is,

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}.$$

In general, we have $D_x \arctan u(x) = \frac{1}{1+u^2} \cdot \frac{du}{dx}$.

EXAMPLE: $D_x \arctan x^3 = \frac{1}{1+x^6} \cdot 3x^2$.

EXAMPLE: $\int \frac{dx}{1+x^2} = \arctan x + C$.

Sample Problem: Evaluate $\int \frac{dx}{4+x^2}$. **Ans:**

$$\int \frac{dx}{4+x^2} = \frac{1}{4}\int \frac{dx}{1+\left(\frac{x}{2}\right)^2} = \frac{1}{2}\arctan\frac{x}{2} + C.$$

8. Arcsine Review

Consider the function $\arcsin x$ also written $\sin^{-1} x$. This means the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ necessary to obtain $\sin x$. The function arcsin is defined for all real numbers between -1 and 1, with those numbers included. It takes its values between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with those values included. If $y = \arcsin x$, then $x = \sin y$. By implicit differentiation, $1 = \cos y \frac{dy}{dx}$, and $\cos y$ is nonnegative for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, so $\cos y = +\sqrt{1-\sin^2 y}$. Therefore, for $-1 < x < 1$ (notice the strict inequality),

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}.$$

This means that on the **open** interval $(-1, 1)$,

$$\int \frac{1}{\sqrt{1-x^2}}\, dx = \arcsin x + C.$$

The function arccos is similar. That is, if $y = \arccos x$, then $x = \cos y$. We fix x with $-1 \le x \le 1$ and y with $0 \le y \le \pi$.

$$\frac{dy}{dx} = \frac{1}{-\sin y} = \frac{1}{-\sqrt{1-\cos^2 y}} = \frac{-1}{\sqrt{1-x^2}}.$$

For integration, stay with arcsin.

EXAMPLES: For $x < 0$, $D_x \arcsin e^x = \frac{e^x}{\sqrt{1-e^{2x}}}$.

Sample Problem: For $-2 < x < 2$, evaluate $\int \frac{1}{\sqrt{4-x^2}}\, dx$. **Ans:**

$$\int \frac{1}{\sqrt{4-x^2}}\, dx = \frac{1}{2}\int \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\, dx = \arcsin\frac{x}{2} + C.$$

We can read off the trig functions of $\arcsin a$, by drawing a right triangle with hypotenuse 1 and opposite side a. Similarly, we can read off the trig functions of $\arctan a$ by drawing a triangle with one leg a and the other 1.

9. TRIGONOMETRIC SUBSTITUTION

Sometimes trig substitutions work as well or better for integration than using inverse trig functions. For example, if for $a > 0$ we have an integral involving $1/\sqrt{a^2 - x^2}$, then we can either use arcsin, or we can try a substitution $x = a\sin\theta$, where we may assume that $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Notice that θ is the new variable. Now

$$\sqrt{a^2 - x^2} = a\sqrt{1 - \sin^2\theta} = a\left|\cos\theta\right| = a\cos\theta$$

since $\cos\theta$ is nonnegative for these values of θ.

Sample Problem: Evaluate $\int_0^1 \frac{dx}{\sqrt{4-x^2}}$ in two ways. **Ans:** Let $x = 2\sin\theta$, so $dx = 2\cos\theta d\theta$. Then

$$\int_0^1 \frac{dx}{\sqrt{4-x^2}} = \int_0^{\pi/6} \frac{2\cos\theta d\theta}{2\cos\theta} = \frac{\pi}{6}.$$

Using arcsin,

$$\int_0^1 \frac{dx}{\sqrt{4-x^2}} = \frac{1}{2}\int_0^1 \frac{dx}{\sqrt{1-\left(\frac{x}{2}\right)^2}} = \left[\arcsin\frac{x}{2}\right]_0^1 = \frac{\pi}{6}.$$

If for $a > 0$ we have an integral involving $a^2 + x^2$, we can try a substitution $x = a\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Now θ is the new variable. Here, x takes all real values. Moreover

$$a^2 + x^2 = a^2\left(1 + \tan^2\theta\right) = a^2\sec^2\theta, \;\; dx = a\sec^2\theta d\theta.$$

Sample Problem For $a = 2$, evaluate $\int_0^{2\sqrt{3}} \frac{dx}{4+x^2}$ in two ways. **Ans:** With the above substitution, when $x = 0$, $\theta = 0$, and when $x = 2\sqrt{3}$, $\theta = \pi/3$. We therefore have

$$\int_0^{2\sqrt{3}} \frac{dx}{4+x^2} = \int_0^{\pi/3} \frac{2\sec^2\theta d\theta}{4\sec^2\theta} = \frac{1}{2}\int_0^{\pi/3} d\theta = \frac{\pi}{6}.$$

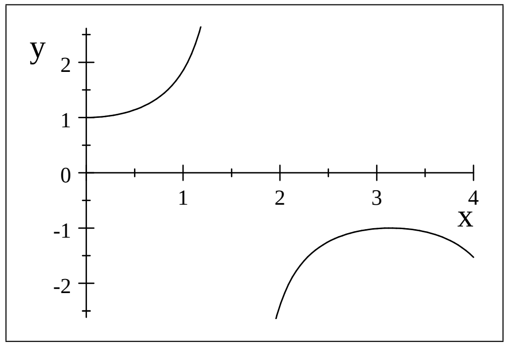
We can also see that

$$\int_0^{2\sqrt{3}} \frac{dx}{4+x^2} = \frac{1}{4}\int_0^{2\sqrt{3}} \frac{dx}{1+\left(\frac{x}{2}\right)^2} = \frac{1}{2}\arctan\left(\frac{x}{2}\right)\Big[_0^{2\sqrt{3}} = \frac{\pi}{6}.$$

10. Arcsec

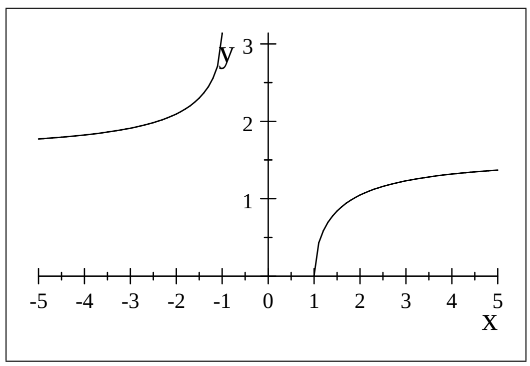
Here is a plot of the function $\sec x$ for $0 \leq x < \pi/2$ and $\pi/2 < x \leq \pi$.

$\sec x$



The function $\operatorname{arcsec} x$, also written $\sec^{-1} x$, is only defined for $|x| \geq 1$. It takes values between 0 and π, with $\frac{\pi}{2}$ excluded.

$\operatorname{arcsec} x$



Set $y = \operatorname{arcsec} x$, so $x = \sec y$. For $x > 1$, $0 < y < \frac{\pi}{2}$, so $\tan y > 0$;of course, $x = \sec y > 0$. For $x < -1$, $\frac{\pi}{2} < y < \pi$, so $\tan y < 0$; of course, $x = \sec y < 0$. It follows that in either case, $\sec y \tan y > 0$, so by implicit differentiation,

$$1 = \sec y \tan y \cdot \frac{dy}{dx} = |\sec y| \sqrt{\sec^2 y - 1} \cdot \frac{dy}{dx}.$$

That is,

$$\frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2 - 1}}.$$

Since

$$D_x \operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2 - 1}},$$

we have the corresponding integral formula

$$\int \frac{1}{|x|\sqrt{x^2 - 1}}\, dx = \operatorname{arcsec} x + C.$$