1. A Tutoring Room is Open

7–9 p.m, Monday, Tuesday, Wednesday, Thursday, Room 140 Lincoln Hall.

2. Homework 27 due Tuesday, December 5 at 9 a.m.

Section 6.7: #4, 6, 12, 18, 26, 28.

Section 6.8: #10, 22, 32, 36, 38, 42.

3. Homework 28 due Thursday, December 7 at 9 a.m.

Section 8.1: #2, 6, 22, 24, 26, 28, 30, 38.

4. Written problem for next week

Here is a problem using the arctan function. A rectangular painting is hung on a wall. How high should the painting be so that the top and bottom subtend the maximum angle for the viewer. We may assume that the viewer is a feet from the picture, and the picture is b feet high. We will also assume that the viewer's eyes are at level $y = 0$, and that the bottom of the painting is at y, which may be positive or negative or 0.

5. Final Exam, Monday December 11, 8-11 a.m.

Section 4 (Liu Qi), Section 5 (Liu Qi) Section 6 (Michael Barrus), Section 8 (Scott Weaver) will take the exam in Room 314 Altgeld Hall.

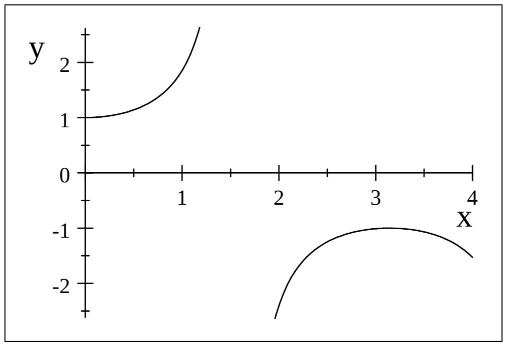
Section 2 (Isaac Goldbring) and half of Section 7 (Isaac Goldbring) (Names beginning with A-G) will take the exam in Room 32 of the Psychology Building. The other half of Section 7 (Isaac Goldbring) and Section 9 (Timothy LeSaulnier) will take the exam in Room 142 of the Psychology Building. People in these sections **must** go to this room and not Altgeld Hall to take the exam.

Everyone should by now know their discussion section and section instructor. You will need to enter that on your examination. Bring your U of I identity card to show when turning in the exam.

6. Arcsec

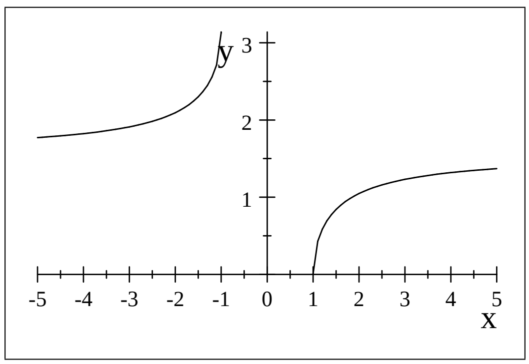
Here is a plot of the function $\sec x$ for $0 \leq x < \pi/2$ and $\pi/2 < x \leq \pi$.

$\sec x$



The function arcsec x, also written $\sec^{-1}x$, is only defined for $|x| \geq 1$. It takes values between 0 and π, with $\frac{\pi}{2}$ excluded.

arcsec x



Set $y = \operatorname{arcsec} x$, so $x = \sec y$. For $x > 1$, $0 < y < \frac{\pi}{2}$, so $\tan y > 0$;of course, $x = \sec y > 0$. For $x < -1$, $\frac{\pi}{2} < y < \pi$, so $\tan y < 0$; of course, $x = \sec y < 0$. It follows that in either case, $\sec y \tan y > 0$, so by implicit differentiation,

$$1 = \sec y \tan y \cdot \frac{dy}{dx} = |\sec y| \sqrt{\sec^2 y - 1} \cdot \frac{dy}{dx}.$$

That is,

$$\frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2 - 1}}.$$

Since

$$D_x \operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2 - 1}},$$

we have the corresponding integral formula

$$\int \frac{1}{|x|\sqrt{x^2 - 1}}\, dx = \operatorname{arcsec} x + C.$$

7. Natural Growth and Decay

For all values of t,

$$\lim_{n\to\infty}(1+\frac{t}{n})^n = e^t.$$

This formula is handy in computing the limit of compound interest as the compounding period goes to 0. For example, suppose one has an initial investment of P_0 dollars at 5% interest compounded once a year. the amount of money at the end of the year is $P_0 \cdot (1+.05)$. On the other hand, if the money is compounded n times during the year, the amount of money at the end of the year is $P_0 \cdot (1+\frac{.05}{n})^n$. If the money is compounded continuously, the amount at the end of the year is $P_0 \cdot e^{.05}$. If the money is left to compound for t years, the amount at the end is $P_0 \cdot (e^{.05})^t = P_0 \cdot e^{.05t}$. This is our first example of exponential growth and decay.

Here is a proof that for this example, the principal $P(t)$ at any time must satisfy the differential equation $dP/dt = \frac{5}{100} \cdot P(t)$. That is, the rate of change at time t is the nominal yearly interest rate multiplied by the amount of money that is present at time t. We will soon see that the solution is what I have given.

Consider a bank account for which the nominal interest rate is $100r\,\%$ per year. For example, if the interest rate is 5 percent, r should be $5/100$. Let us only assume that money is put into the account at this rate in such a way that the principal $P(t)$ is a **continuous** function of the time t in years. For a time interval $[t_0, t_0+\Delta t]$, the change ΔP must be a function of Δt with limit 0 as $\Delta t \to 0$. Moreover, we must have

$$r\cdot\Delta t\cdot P(t_0) \le \Delta P \le r\cdot\Delta t\cdot(P(t_0)+\Delta P)\,.$$

Dividing by Δt and taking the limit as $\Delta t \to 0+$, it follows that

$$r\cdot P(t_0) \le \lim_{\Delta t\to 0+}\frac{\Delta P}{\Delta t} \le r\cdot P(t_0) + r\cdot\lim_{\Delta t\to 0+}\Delta P = r\cdot P(t_0).$$

A similar result holds for negative values of Δt, and so the function $P(t)$, if it exists, must be a differentiable function of t satisfying the equation $dP/dt = r\cdot P(t)$ at any time t.

Continuously compounding interest and radioactive decay are examples of functions with the property that the rate of change $dQ(t)/dt$ is a constant k times the amount $Q(t)$ present at the time. If $k > 0$, then $Q(t)$ is increasing, while if $k < 0$, $Q(t)$ is decreasing. If $k = 0$, we have a constant function. In any case, the solution to this differential equation is $Q(t) = Q_0e^{kt}$, where Q_0 is the amount at time 0.

In many problems, you have to find either Q_0, k, or t. For example, $k = Q'(t)/Q(t)$ at any time t. Also, if for some positive number a you want the value of t such that $a = Q_0e^{kt}$, then taking the natural logarithm of both sides of this equation, we have

$$\ln a = \ln Q_0 + kt, \quad \text{so} \quad t = \frac{\ln a - \ln Q_0}{k}.$$

Note that if $k > 0$, then we must have $a \geq Q_0$ to solve this for a nonnegative value of t; if $k < 0$, we must have $a \leq Q_0$.