

1. A TUTORING ROOM IS OPEN

7–9 p.m, Monday, Tuesday, Wednesday, Thursday, Room 140 Lincoln Hall.

2. HOMEWORK 27 DUE TUESDAY, DECEMBER 5 AT 9 A.M.

Section 6.7: #4, 6, 12, 18, 26, 28.

Section 6.8: #10, 22, 32, 36, 38, 42.

3. HOMEWORK 28 DUE THURSDAY, DECEMBER 7 AT 9 A.M.

Section 8.1: #2, 6, 22, 24, 26, 28, 30, 38.

4. WRITTEN PROBLEM FOR THIS WEEK

Here is a problem using the arctan function. A rectangular painting is hung on a wall. How high should the painting be so that the top and bottom subtend the maximum angle for the viewer. We may assume that the viewer is a feet from the picture, and the picture is b feet high. We will also assume that the viewer's eyes are at level $y = 0$, and that the bottom of the painting is at y , which may be positive or negative or 0.

5. FINAL EXAM, MONDAY DECEMBER 11, 8-11 A.M.

Section 4 (Liu Qi), Section 5 (Liu Qi) Section 6 (Michael Barrus), Section 8 (Scott Weaver) will take the exam in Room 314 Altgeld Hall.

Section 2 (Isaac Goldbring) and half of Section 7 (Isaac Goldbring) (Names beginning with A-G) will take the exam in Room 32 of the Psychology Building. The other half of Section 7 (Isaac Goldbring) and Section 9 (Timothy LeSaulnier) will take the exam in Room 142 of the Psychology Building. People in these sections **must** go to this room and not Altgeld Hall to take the exam.

Everyone should by now know their discussion section and section instructor. You will need to enter that on your examination. Bring your U of I identity card to show when turning in the exam.

6. MORE ON NATURAL GROWTH AND DECAY

Continuously compounding interest and radioactive decay are examples of functions with the property that the rate of change $dQ(t)/dt$ is a constant k times the amount $Q(t)$ present at the time. If $k > 0$, then $Q(t)$ is increasing, while if $k < 0$, $Q(t)$ is decreasing. If $k = 0$, we have a constant function. In any case, the solution to this differential equation is $Q(t) = Q_0 e^{kt}$, where Q_0 is the amount at time 0.

In many problems, you have to find either Q_0 , k , or t . For example, $k = Q'(t)/Q(t)$ at any time t . Also, if for some positive number a you want the value of t such that $a = Q_0 e^{kt}$, then taking the natural logarithm of both sides of this equation, we have

$$\ln a = \ln Q_0 + kt, \quad \text{so} \quad t = \frac{\ln a - \ln Q_0}{k}.$$

Note that if $k > 0$, then we must have $a \geq Q_0$ to solve this for a nonnegative value of t ; if $k < 0$, we must have $a \leq Q_0$.

Here is a precise statement of the fact we are using.

Theorem 1. *Let $Q(t)$ be an unknown function of time t such that for some constant k , the derivative $Q'(t) = k \cdot Q(t)$ and $Q(0) = Q_0$. Then we must have $Q(t) = Q_0 e^{kt}$.*

Proof. Consider the function $Q(t) = Q_0 e^{kt}$. Taking the derivative, we have

$$Q'(t) = D_t (Q_0 e^{kt}) = k \cdot Q_0 e^{kt} = kQ(t)$$

and $Q_0 = Q(0)$, so this function works. What we now have to prove is that this is the only function that works. Suppose $f(t)$ is a function of time such that $f'(t) = k \cdot f(t)$ and $f(0) = Q_0$. Consider the function $f(t)e^{-kt}$. Differentiating, we get

$$f'(t)e^{-kt} - kf(t)e^{-kt} = kf(t)e^{-kt} - kf(t)e^{-kt} \equiv 0.$$

Therefore, for some constant C , $f(t)e^{-kt} = C$. At $t = 0$, we have $Q_0 = C$, so $f(t)e^{-kt} = Q_0$, or $f(t) = Q_0 e^{kt}$. Therefore $Q(t) = Q_0 e^{kt}$ is the only function that works. \square

Sample Problem: We have seen that if \$100 is put in the bank at 5% and left for 20 years, the amount of money in the account at the end of 20 years is $\$100 \cdot (1 + \frac{5}{100})^{20} = \265.33 . How much more would be in the account at the end of 20 years if the money had compounded continuously? **Ans.** In this case, there would be $\$100e^{\frac{5}{100} \cdot 20} = \$100e = \$271.83$, that is \$6.50 more.

Sample Problem: Suppose a radioactive material decays at a rate given by a negative constant $-k$. That is the amount of material at time t is $Q(t) = Q_0 e^{-kt}$. What is the half life of this material?

Ans: Here we want the value of t such that $Q(t) = \frac{1}{2}Q_0$, that is, $e^{-kt} = \frac{1}{2}$. Taking natural logarithms of both sides of this equation we have $-kt = -\ln 2$, or $t = \frac{\ln 2}{k}$.

7. RADIOCARBON DATING

Radioactive Carbon 14 has a half life of 5700 years. That is,

$$\frac{1}{2} = e^{-k5700}, \text{ or } -\ln 2 = -k \cdot 5700, \text{ or } k = 0.0001216.$$

During the life of an organism, the ratio of radioactive Carbon 14 to normal Carbon is the same as the air. After death, the Carbon 14 decays and is not replaced. Therefore,

the amount of Carbon 14 that is present compared to the amount of normal Carbon gives a good estimate of the time the organism died.

EXAMPLE: Charcoal found at Stonehenge contains 63% as much Carbon 14 as a sample of charcoal produced from a newly felled tree. The is, if t is the number of years between the time the tree producing that charcoal was cut down and the present time, then $e^{-kt} = .63$, where we have seen that $k = 0.0001216$. Therefore, the charcoal at Stonehenge was produced approximately 3800 years ago.

8. REVIEW

Sample Problem: Find the equation of the line through the points $(-2, 1)$ and $(3, -5)$. **Ans.** The slope of the line is $(-5 - 1)/(3 - (-2)) = -6/5$. The equation of the line is $y - 1 = (-6/5)(x + 2)$.

Sample Problem: Complete the square to derive the quadratic formula starting from the equation $ax^2 + bx + c = 0$ with $a > 0$. **Ans:** One writes the equivalent equation $x^2 + \frac{b}{a}x = -\frac{c}{a}$, then completes the square, to get

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}, \text{ or } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Sample Problem: Show that $\lim_{x \rightarrow 2} x^2 + 5x - 7 = 7$ **using the facts** that $\lim_{h \rightarrow 0} h^2 = 0$ and $\lim_{h \rightarrow 0} 9h = 0$. **Ans:**

$$\begin{aligned} \lim_{h \rightarrow 0} [(2 + h)^2 + 5(2 + h) - 14] &= \lim_{h \rightarrow 0} [4 + 9h + h^2 - 4] \\ &= \lim_{h \rightarrow 0} [9h + h^2] = 0. \end{aligned}$$

Sample Problem: Find the equations of the tangent line and normal line to the parabola $y = 3x^2 + 2x + 1$ at the point $(1, 6)$. **Ans:** Let $f(x) = 3x^2 + 2x + 1$. Then $f'(x) = 6x + 2$ at each x . At $x = 1$, the tangent line has slope 8. That is, if we now let x and y denote the variables for the tangent line, $\frac{y-6}{x-1} = 8$, i.e., $y = 8x - 8 + 6 = 8x - 2$ is the equation for the tangent line to the graph of f at $(1, 6)$. One check is to make sure the tangent line we describe has the right slope 8 and goes through the point $(1, 6)$. The normal line has slope $-1/8$, so the equation is $y - 6 = (-1/8)(x - 1)$.

Sample Problem: Show that $f(h) = h \cdot \cos(h)$ has limit 0 at 0. **Ans:** Since $|\cos(h)| \leq 1$ for all h and $\lim_{h \rightarrow 0} h = 0$, $\lim_{h \rightarrow 0} h \cdot \cos(h) = 0$. Alternatively,

$$-h \leq h \cdot \cos(h) \leq h, \quad \text{and} \quad \lim_{h \rightarrow 0} h = \lim_{h \rightarrow 0} -h = 0,$$

so by the squeeze law $\lim_{h \rightarrow 0} h \cdot \cos(h) = 0$.