

1. A TUTORING ROOM IS OPEN

7–9 p.m, Monday, Tuesday, Wednesday, Thursday, Room 140 Lincoln Hall.

2. HOMEWORK 28 DUE THURSDAY, DECEMBER 7 AT 9 A.M.

Section 8.1: #2, 6, 22, 24, 26, 28, 30, 38.

3. WRITTEN PROBLEM FOR THIS WEEK

Here is a problem using the arctan function. A rectangular painting is hung on a wall. How high should the painting be so that the top and bottom subtend the maximum angle for the viewer. We may assume that the viewer is a feet from the picture, and the picture is b feet high. We will also assume that the viewer's eyes are at level $y = 0$, and that the bottom of the painting is at y , which may be positive or negative or 0.

4. FINAL EXAM, MONDAY DECEMBER 11, 8-11 A.M.

Section 4 (Liu Qi), Section 5 (Liu Qi) Section 6 (Michael Barrus), Section 8 (Scott Weaver) will take the exam in Room 314 Altgeld Hall.

Section 2 (Isaac Goldbring) and half of Section 7 (Isaac Goldbring) (Names beginning with A-G) will take the exam in Room 32 of the Psychology Building. The other half of Section 7 (Isaac Goldbring) and Section 9 (Timothy LeSaulnier) will take the exam in Room 142 of the Psychology Building. People in these sections **must** go to this room and not Altgeld Hall to take the exam.

Everyone should by now know their discussion section and section instructor. You will need to enter that on your examination. Bring your U of I identity card to show when turning in the exam.

5. REVIEW

Sample Problem: Find the equation of the line through the points $(-2, 1)$ and $(3, -5)$. **Ans.** The slope of the line is $(-5 - 1)/(3 - (-2)) = -6/5$. The equation of the line is $y - 1 = (-6/5)(x + 2)$.

Sample Problem: Complete the square to derive the quadratic formula starting from the equation $ax^2 + bx + c = 0$ with $a > 0$. **Ans:** One writes the equivalent equation $x^2 + \frac{b}{a}x = -\frac{c}{a}$, then completes the square, to get

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}, \text{ or } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Sample Problem: Show that $\lim_{x \rightarrow 2} x^2 + 5x - 7 = 7$ **using the facts** that $\lim_{h \rightarrow 0} h^2 = 0$ and $\lim_{h \rightarrow 0} 9h = 0$. **Ans:**

$$\begin{aligned} \lim_{h \rightarrow 0} [(2+h)^2 + 5(2+h) - 14] &= \lim_{h \rightarrow 0} [4 + 9h + h^2 - 4] \\ &= \lim_{h \rightarrow 0} [9h + h^2] = 0. \end{aligned}$$

Sample Problem: Find the equations of the tangent line and normal line to the parabola $y = 3x^2 + 2x + 1$ at the point $(1, 6)$. **Ans:** Let $f(x) = 3x^2 + 2x + 1$. Then $f'(x) = 6x + 2$ at each x . At $x = 1$, the tangent line has slope 8. That is, if we now let x and y denote the variables for the tangent line, $\frac{y-6}{x-1} = 8$, i.e., $y = 8x - 8 + 6 = 8x - 2$ is the equation for the tangent line to the graph of f at $(1, 6)$. One check is to make sure the tangent line we describe has the right slope 8 and goes through the point $(1, 6)$. The normal line has slope $-1/8$, so the equation is $y - 6 = (-1/8)(x - 1)$.

Sample Problem: Show that $f(h) = h \cdot \cos(h)$ has limit 0 at 0. **Ans:** Since $|\cos(h)| \leq 1$ for all h and $\lim_{h \rightarrow 0} h = 0$, $\lim_{h \rightarrow 0} h \cdot \cos(h) = 0$. Alternatively,

$$-h \leq h \cdot \cos(h) \leq h, \quad \text{and} \quad \lim_{h \rightarrow 0} h = \lim_{h \rightarrow 0} -h = 0,$$

so by the squeeze law $\lim_{h \rightarrow 0} h \cdot \cos(h) = 0$.

EXAM 1

1a) Find the largest possible domain for the function $f(x) = \sqrt{x^2 - 36}$.

Ans: $x \leq -6$ or $x \geq 6$. Do **NOT** write $6 \leq x \leq -6$; 6 is not smaller than -6 .

1b) If $y = f(x) = x^3 - 6x + 5$, at what values of x is the tangent line to the graph horizontal?

Ans: $f'(x) = 3x^2 - 6 = 0$ when $x = \pm\sqrt{2}$.

1c) Write the equation of the line through the point $(-3, 2)$ parallel to the line $2y + 8x - 5 = 0$.

Ans: The line is given by $y = -4x + 5/2$; it has slope -4 . The desired line is given by $y - 2 = -4(x + 3)$, or $y = -4x - 10$. To check, we see that this is a line with slope -4 and it goes through the point $(-3, 2)$.

2a) *Find the center and radius of the circle $x^2 + y^2 - 4y - 5 = 0$. **Ans:** Completing the square, we have $x^2 + (y - 2)^2 - 4 - 5 = 0$. this is the equation of a circle with center $(0, 2)$ and radius 3.

2b) *Find $\lim_{x \rightarrow 16} \frac{x-16}{\sqrt{x}-4}$. Hint: $x - 16 = (\sqrt{x} - 4)(\sqrt{x} + 4)$.

$$\text{Ans: } \lim_{x \rightarrow 16} \frac{x-16}{\sqrt{x}-4} = \lim_{x \rightarrow 16} (\sqrt{x} + 4) = 8.$$

3a) *Find $\lim_{x \rightarrow 0} \frac{\sin^2 4x}{7x^2}$.

$$\text{Ans: } \lim_{x \rightarrow 0} \frac{\sin^2 4x}{7x^2} = \lim_{x \rightarrow 0} \left(\frac{16}{7} \cdot \frac{\sin 4x}{4x} \cdot \frac{\sin 4x}{4x} \right) = \frac{16}{7}.$$

3b) *Let $f(x) = \frac{3 \sin x}{x}$ for $x < 0$, and let $f(x) = \frac{1 - \cos x}{x} + A$ for $x > 0$. Find a value for A so that 0 is a removable discontinuity of f .

$$\text{Ans: } \lim_{x \rightarrow 0^-} \frac{3 \sin x}{x} = 3 \text{ and } \lim_{x \rightarrow 0^+} \frac{1 - \cos x}{x} = 0, \text{ so we must set } A = 3.$$

4a) *There is a result that tells you that the equation $f(x) = x^5 - x^2 + 5x - 2 = 0$ has a solution in the interval between -1 and 1 . Give the name of that result, or at least describe what it says. (Hint: What is $f(-1)$; what is $f(1)$?) Ans: $f(-1) = -9$, and $f(1) = 3$. Since f is continuous, it must take the value 0 someplace between -1 and 1 by the Intermediate Value Theorem.

4b) *Let $f(x) = x^2 + 2x - 8$. We know that $\lim_{x \rightarrow 3} f(x) = 7$, and this means that $f(3 + h) = 7 + E(h)$ where $\lim_{h \rightarrow 0} E(h) = \lim_{h \rightarrow 0} (f(3 + h) - 7) = 0$. Write the equation expressing $E(h)$ as a function of h for this case. Ans: Note, your answer should be a polynomial in h .

$$\begin{aligned} f(3 + h) &= (3 + h)^2 + 2(3 + h) - 8 \\ &= 9 + 6h + h^2 + 6 + 2h - 8 \\ &= 7 + (8h + h^2). \end{aligned}$$

$$\text{so } E(h) = 8h + h^2.$$

5a) Use the **product rule** to find the following derivative, show all your work and **do not simplify**: Here is the answer:

$$\begin{aligned} D_x (3x^5 - 2x^2)(3x^4 + 7x + 2) \\ = (15x^4 - 4x)(3x^4 + 7x + 2) + (3x^5 - 2x^2)(12x^3 + 7). \end{aligned}$$

5b) Find the equation of the tangent line to the graph of the function given by the equation $y = x^3 - 3x^2 + 2x + 2$ at the point $(1, 2)$.

Ans: $dy/dx = 3x^2 - 6x + 2$. at $x = 1$, this is $3 - 6 + 2 = -1$. The equation of the tangent line is $y - 2 = -(x - 1)$, or $y = -x + 3$. As a check, this is the equation of the line with slope -1 through the point $(1, 2)$.

6a) *The limit $\lim_{\Delta x \rightarrow 0} \frac{\sqrt{2+\Delta x} - \sqrt{2}}{\Delta x}$ is the derivative of what function at what point?

Ans: The function is $f(x) = \sqrt{x}$, the point is $x = 2$.

6b) *Let f be a function with a derivative $f'(a)$ at a . We have seen that this means that for small nonzero values for Δx ,

$$\frac{\Delta y}{\Delta x} = \frac{f(a + \Delta x) - f(a)}{\Delta x} = f'(a) + E(\Delta x)$$

where $\lim_{\Delta x \rightarrow 0} E(\Delta x) = 0$. Multiplying by Δx we get

$$\Delta y = f'(a) \cdot \Delta x + E(\Delta x) \cdot \Delta x.$$

Why does the last equation tell us that f is continuous at a ?

Ans: The limit of Δy is 0 as Δx goes to 0.

7) *Let $y = f(x) = |x|$. Let us try to calculate a derivative at $x = 0$. For $\Delta x > 0$, $\Delta y = |0 + \Delta x| - 0 = \Delta x$. For $\Delta x < 0$, $\Delta y = |0 + \Delta x| - 0 = -\Delta x$.

a) What is $\lim_{\Delta x \rightarrow 0^+} \frac{\Delta y}{\Delta x}$? Ans: 1.

b) What is $\lim_{\Delta x \rightarrow 0^-} \frac{\Delta y}{\Delta x}$? Ans: -1 .

c) Why does the derivative of $f(x) = |x|$ not exist at $x = 0$? Ans: because the left and right hand limit of $\frac{\Delta y}{\Delta x}$ are not the same.

Sample Problem: Find $\lim_{x \rightarrow 9} \frac{x-9}{\sqrt{x}-3}$. **Ans:**

$$\lim_{x \rightarrow 9} \frac{x-9}{\sqrt{x}-3} = \lim_{x \rightarrow 9} \frac{\sqrt{x}-3}{\sqrt{x}-3} \cdot \frac{\sqrt{x}+3}{1} = \lim_{x \rightarrow 9} (\sqrt{x}+3) = 6.$$

Sample Problem: Use the definition of the derivative (a.k.a., slope predictor function) to find the derivative of $f(x) = x^3$ at every x . **Do not** just invoke the power rule; show all your work. **Ans:**

$$\begin{aligned} \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x} \\ &= 3x^2 + \lim_{\Delta x \rightarrow 0} (3x\Delta x + (\Delta x)^2) = 3x^2. \end{aligned}$$

Sample Problem: Evaluate $\lim_{x \rightarrow 0} \frac{\sin 5x}{\sin 3x}$ and $\lim_{x \rightarrow 0} \frac{5x}{\tan 3x}$. **Ans:**

$$\lim_{x \rightarrow 0} \frac{\sin 5x}{\sin 3x} = \frac{5}{3} \cdot \lim_{x \rightarrow 0} \left(\frac{3x}{\sin 3x} \cdot \frac{\sin 5x}{5x} \right) = \frac{5}{3} \cdot \lim_{x \rightarrow 0} \left(\frac{3x}{\sin 3x} \right) \cdot \lim_{x \rightarrow 0} \left(\frac{\sin 5x}{5x} \right) = \frac{5}{3}.$$

$$\lim_{x \rightarrow 0} \frac{5x}{\tan 3x} = \frac{5}{3} \lim_{x \rightarrow 0} \frac{3x}{\tan 3x} = \frac{5}{3} \cdot \lim_{x \rightarrow 0} \frac{3x}{\sin 3x} \cdot \lim_{x \rightarrow 0} \cos(3x) = \frac{5}{3}.$$

Sample Problem: Let $f(x) = (x^2 - 4)/(x - 2)$ for $x \neq 2$, and let $f(2) = a$. For what value of a is f continuous at 2? **Ans:** $\lim_{x \rightarrow 2} f(x) = 4$, so we must have $a = 4$.

Sample Problem: Show that \sin and \cos are each continuous functions at any point a . **Ans:** We note that for small values of h including $h = 0$,

$$\begin{aligned} \sin(a + h) &= \sin a \cos h + \cos a \sin h, \\ \cos(a + h) &= \cos a \cos h - \sin a \sin h. \end{aligned}$$

Since $\lim_{h \rightarrow 0} \cos h = 1$, and $\lim_{h \rightarrow 0} \sin h = 0$, $\lim_{h \rightarrow 0} \sin(a+h) = \sin a$ and $\lim_{h \rightarrow 0} \cos(a+h) = \cos a$.

Sample Problem: Show that the function $f(x) = x^5 - 3x^3 + 1$ takes the value 0 somewhere in the interval $(0, 1)$. **Ans:** $f(0) = 1$, and $f(1) = -1$, so by the Intermediate Value Property, there is a point c in the open interval $(0, 1)$ where $f(c) = 0$.

Sample Problem: Evaluate the following derivative, show all your work and **do not simplify**.

$$D_x \frac{5x^3 - 2x + 7}{x^2 - 7x + 2}.$$

Ans:

$$D_x \frac{5x^3 - 2x + 7}{x^2 - 7x + 2} = \frac{(15x^2 - 2x)(x^2 - 7x + 2) - (5x^3 - 2x + 7)(2x - 7)}{(x^2 - 7x + 2)^2}$$

Sample Question: Using the Chain Rule, evaluate $D_x \frac{1}{(x^2 - 5x + 2)^3}$. Here is the answer:

$$\begin{aligned} D_x \frac{1}{(x^2 - 5x + 2)^3} &= D_x (x^2 - 5x + 2)^{-3} = -3(x^2 - 5x + 2)^{-4} \cdot (2x - 5) \\ &= -\frac{3}{(x^2 - 5x + 2)^4} (2x - 5). \end{aligned}$$

Sample Question: Let V be the volume of a sphere. Suppose that $dV/dt = 5 \text{ cm}^3/\text{sec}$ at all times t . What is dr/dt when $r = 2 \text{ cm}$? Here is the answer: Since $\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$

$$\frac{dr}{dt} = \frac{\frac{dV}{dt}}{\frac{dV}{dr}} = \frac{5}{4\pi r^2} = \frac{5}{16\pi} \text{ cm/sec}.$$

Note: Do not apply the fact that at the instant in question $r = 2$ until you have differentiated. If $r = 2$ at all times, then V is constant and has 0 derivative.

Sample Question: Let $y = x^{2/3}$. Use the fact that $y^3 = x^2$ to show that $dy/dx = (2/3)x^{-1/3}$. Here is the answer:

$$\begin{aligned} D_x y^3 &= 3y^2 \frac{dy}{dx} = 2x \\ \frac{dy}{dx} &= \frac{2x}{3y^2} = \frac{2x}{3x^{4/3}} = \frac{2}{3}x^{-1/3}. \end{aligned}$$

Sample Question: A large cube of ice is decreasing in volume at a rate of 6 cm^3 per minute. Find the rate of change of the edge (it should be negative) when the volume is 64 cm^3 ? Show all your work. **Ans:** Let $x(t)$ be the length of the edge at time t . Since the volume $V = x^3$, $-6 = dV/dt = 3x^2 dx/dt$. When the volume is 64 , $x = 4$, and then $dx/dt = -6/(3 \cdot 16) = -1/8 \text{ cm/minute}$.

Sample Question: Assume f is defined and continuous on an interval I . What are the **critical points** (or numbers) of f on I ? **Ans:** They are listed as follows:

- 1) Endpoints of I if they belong to I .
- 2) Points of I where the derivative of f does not exist.
- 3) Points of I where the derivative exists and is 0.

Sample Question: Let $f(x) = x^3 - 3x^2 + 3x - 2$ on the interval $[0, 2]$. List all critical points of f on this interval, and find the maximum and minimum values of $f(x)$ on this interval and points of inflection.

SOLUTION. The derivative f' exists at all points and has the value $3x^2 - 6x + 3$. To find where $f'(x) = 0$, we divide by 3 and solve the equation $x^2 - 2x + 1 = 0$. The only solution is $x = 1$. The critical points are $0, 1, 2$. Now $f(0) = -2$, $f(1) = -1$, $f(2) = 8 - 12 + 6 - 2 = 0$. Thus, for the interval $[0, 2]$, f takes an absolute minimum value -2 at 0 and f takes an absolute maximum value 0 at 2 . Since $f''(x) = 6x - 6 < 0$ for $x < 1$ and $f''(x) > 0$ for $x > 1$, 1 is a point of inflection for f .

Sample Problem: Find $D_x \sin \sqrt{x}$.

Solution:

$$D_x \sin \sqrt{x} = (\cos \sqrt{x}) \cdot \frac{1}{2\sqrt{x}}.$$

Sample Problem: Simplify $\log_{10} \frac{5000}{20}$. **Ans:**

$$\begin{aligned} \log_{10} \frac{5000}{20} &= \log_{10} 5000 - \log_{10} 20 \\ &= \log_{10} 5 + \log_{10} 1000 - \log_{10} 2 - \log_{10} 10 = 2 + \log_{10} 2.5. \end{aligned}$$

Sample Question: Find the derivative of $e^{\sin x}$. **Ans:**

$$D_x e^{\sin x} = e^{\sin x} \cdot \cos x.$$

Sample Question: For what special value of a is $\lim_{\Delta x \rightarrow 0} \frac{a^{\Delta x} - 1}{\Delta x} = 1$? **Ans:**
 $a = e$.

Sample Question: Use logarithmic differentiation to find dy/dx when $y = x^{\sin x}$.
Ans: Taking natural log of both sides we get $\ln y = \sin x \cdot \ln x$, so

$$\frac{dy}{dx} = y \left(\cos x \cdot \ln x + \frac{\sin x}{x} \right) = x^{\sin x} \left(\cos x \cdot \ln x + \frac{\sin x}{x} \right).$$

Sample Question: Write the equation of the tangent line to the graph of the curve $\frac{1}{1+x} + \frac{1}{1+y} = x$ at the point $(1, 1)$.

SOLUTION: Rather than solve for y as a function of x , we use implicit differentiation. That is, we think of y as a function of x and differentiate both sides of the equation with respect to x . This gives us

$$\frac{-1}{(1+x)^2} + \frac{-1}{(1+y)^2} \cdot \frac{dy}{dx} = 1.$$

Now we substitute the values for x and y and get $\frac{-1}{4} + \frac{-1}{4} \cdot \frac{dy}{dx} = 1$. That is, $\frac{dy}{dx} = \frac{5/4}{-1/4} = -5$. We are looking for the equation of the line with slope -5 going through the point $(1, 1)$. Now using x and y for the variables of the line, we get $y - 1 = -5(x - 1)$, or $y = -5x + 6$.

Sample Problem A ladder 41 feet long is leaning against a vertical wall. The ladder begins to slip. The ladder's top slides down the wall while its bottom moves along the level ground at a constant speed of 10 ft/sec. How fast is the top of the ladder moving when it is 9 ft above the ground?

SOLUTION: Let x be the distance of the foot of the ladder from the wall and y be the distance of the top from the ground. Both are functions of time t . We are given that $x^2 + y^2 = 41^2$. It follows that $2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$, so $\frac{dy}{dt} = \frac{-x}{y} \cdot \frac{dx}{dt}$. At the time in question, $y = 9$, $x = \sqrt{41^2 - 9^2} = 40$, and $\frac{dx}{dt} = +10$. Therefore, $\frac{dy}{dt} = -\frac{400}{9}$

feet/second. The minus is because the distance of the top of the ladder from the ground is decreasing. Notice we do not put in the value $y = 9$ until after we have differentiated.

Sample Problem: Given a differentiable function $y = f(x)$, you want to find a value x such that $f(x) = 0$. You make a guess x_1 such that the derivative $f'(x_1) \neq 0$. **a)** Write the formula for the tangent line L at the point $(x_1, f(x_1))$. **b)** Your next guess x_2 is the x -coordinate of the point where that tangent line L intersects what line?

Ans: a) In terms of the variables x and y for the line, $y - f(x_1) = f'(x_1)(x - x_1)$.

Ans: b) The x -axis.

Sample Question: For $y = x^3$ and a small change Δx , the change in y , that is Δy , is the derivative $3x^2$ multiplied by Δx plus a function $E(\Delta x)$ multiplied Δx where $\lim_{\Delta x \rightarrow 0} E(\Delta x) = 0$. What is $E(\Delta x)$ in this case. **Ans:** $E(\Delta x)$ is $3x \cdot \Delta x + (\Delta x)^2$.

Sample Problem Use differentials, that is, a linear approximation, to estimate $30^{1/5}$. We know that $2^5 = 32$, whence $32^{1/5} = 2$. Let $y = x^{1/5}$. Then $dy = \frac{1}{5}x^{-4/5}dx$. Working at the point $x = 32$, $dx = -2$ with the formula $y + \Delta y \approx y + dy = y + \frac{dy}{dx}dx$, we have

$$30^{1/5} \approx 32^{1/5} + \frac{1}{5}(32)^{-4/5}(-2) = 1.975.$$

The actual value is 1.9744.

Sample Problem: Suppose $y = f(x)$ is a differentiable function that is not always 0 on the interval $[2, 7]$ but $f(2) = f(7) = 0$. Must the derivative f' take the value 0 at some point between 2 and 7? Why? **Ans:** Yes, because f is continuous, $f(x)$ must either take a positive maximum value or a negative minimum value at some point between 2 and 7. at such a point c , $f'(c) = 0$.

Sample Problem: Suppose f is a differentiable function on the real line and the derivative $f'(x) \geq 9$ for all x . Is it possible that $f(1) = 5$ and $f(2) = 8$? Explain.

Ans. The answer is no, $(f(2) - f(1)) / (2 - 1) = 3$, but there is no point c between 1 and 2 where $f'(c) = 3$.

Sample Problem: Suppose f is a function on the real line such that $|f'(x)| \leq 1$ for all real numbers x . Use the Mean Value Theorem to show that for any two numbers x_1 and x_2 , we must have

$$|f(x_1) - f(x_2)| \leq |x_1 - x_2|.$$

Ans: By the Mean Value Theorem there is a point c between x_1 and x_2 such that

$$|f(x_1) - f(x_2)| = |f'(c)| |x_1 - x_2| \leq |x_1 - x_2|.$$

Sample Problem: You have a canal that makes a right angle turn. The width of the canal for the incoming leg is a , and the width of the canal for the outgoing leg is b . What is the longest narrow barge that can be moved around the turn?

Ans: Let θ be the angle formed by a line segment touching the inside corner and terminating at the outside walls of the canal; here the angle is formed by the line segment and the outside wall of the leg of width b . The length of the line segment is

$$L(\theta) = a \sec \theta + b \csc \theta.$$

As θ approaches 0 or $\frac{\pi}{2}$, the length of this line segment approaches $+\infty$. Therefore, the maximum length of a barge that can make the turn is the minimum value of $L(\theta)$, which is at a value θ where $L'(\theta) = 0$. Now we make the following calculations to find the desired value of θ :

$$\begin{aligned} L'(\theta) &= a \sec \theta \tan \theta - b \csc \theta \cot \theta = 0 \\ a \sec \theta \tan \theta &= b \csc \theta \cot \theta \\ \frac{\sec \theta \tan \theta}{\csc \theta \cot \theta} &= \frac{\sin \theta}{\cos \theta} \cdot \tan^2 \theta = \tan^3 \theta = \frac{b}{a}, \end{aligned}$$

or $\tan \theta = \frac{b^{1/3}}{a^{1/3}}$. Now we consider a right triangle with the side adjacent to the angle θ having length $a^{1/3}$, while the side opposite θ has length $b^{1/3}$. This gives the right value for $\tan \theta$, and the hypotenuse has length $\sqrt{a^{2/3} + b^{2/3}}$. Using this triangle, we see that the length

$$\begin{aligned} L(\theta) &= a \sec \theta + b \csc \theta \\ &= a \frac{\sqrt{a^{2/3} + b^{2/3}}}{a^{1/3}} + b \frac{\sqrt{a^{2/3} + b^{2/3}}}{b^{1/3}} \\ &= (a^{2/3} + b^{2/3}) \sqrt{a^{2/3} + b^{2/3}} = (a^{2/3} + b^{2/3})^{3/2}. \end{aligned}$$

Sample Problem: Consider the equation $f(x) = x^3 + 9x^2 + 33x - 8 = 0$; show that it has exactly one real solution. You may use Rolle's Theorem and the fact that $f(-1) < 0$, and $f(1) > 0$. **Ans:** Now $f(-1) < 0$, and $f(1) > 0$, so by the Intermediate Value Theorem, the graph of the function must cross the x axis between -1 and 1 . On the other hand, $f'(x) = 3x^2 + 18x + 33 = 0$ when $x^2 + 6x + 11 = 0$. There are no real solutions for the last equation. Therefore, by Rolle's theorem, there can only be one point where $f(x) = 0$.

6. EXAM 2

1) Find the following derivatives. Show all your work and **do not simplify**.

a) $D_x \left(\frac{3x^2 - 5x + 2}{4 + \sin x} \right) = \frac{(6x - 5)(4 + \sin x) - (3x^2 - 5x + 2)(\cos x)}{(4 + \sin x)^2}$.

1b) $D_x (\ln(x^2 + 7) \cdot \sec(x^3 - 3)) = \frac{2x}{x^2 + 7} \cdot \sec(x^3 - 3) + \ln(x^2 + 7) \cdot \sec(x^3 - 3) \cdot \tan(x^3 - 3) \cdot 3x^2$.

1c) $D_x (\cos^5(e^{x^2})) = -5 \cos^4(e^{x^2}) (\sin(e^{x^2})) (e^{x^2}) (2x)$.

2 a) Write the equation of the tangent line to the graph of the curve given by the equation $\frac{1}{1+y} + \ln x = \frac{x}{2}$ at the point $(1, 1)$. **Ans:** Using implicit differentiation, we have

$$\frac{-1}{(1+y)^2} \frac{dy}{dx} + \frac{1}{x} = \frac{1}{2}.$$

At the point $(1, 1)$ this is $(-1/4) dy/dx = -1/2$, or $dy/dx = 2$. The equation of the tangent line is $y - 1 = 2(x - 1)$, or $y = 2x - 1$.

2 b) The position function of a particle moving in a horizontal straight line with motion to the right being positive is given by $x(t) = -2t^3 - 3t^2 + 12t + 5$ feet for all times $t \geq 0$, where the time t is given in seconds. Find the velocity and the acceleration of the particle at any nonnegative time t , and find the particle's position at a positive time when its velocity is zero. **Ans:** The velocity $v(t) = -6t^2 - 6t + 12$ feet/sec, and the acceleration $a(t) = -12t - 6$ feet/sec/sec. The velocity $v(t) = 0$ at a positive time when $t = 1$, and at that time $x(1) = 12$ feet.

3) Let $f(x) = 2x^3 - 3x^2 + 6$ on the interval $[0, 2]$. List all critical points of f on this interval, and find the maximum and minimum values of $f(x)$ on this interval. **Ans:** $f'(x) = 6x^2 - 6x = 0$ when $x = 0$ and $x = 1$. The critical points are 0, 1, and 2. $f(0) = 6$, $f(1) = 5$, and $f(2) = 10$. The minimum value of $f(x)$ on $[0, 2]$ is 5 and the maximum value is 10.

4) Let $f(x) = x$ and $g(x) = x^5$. Find the value of x in the interval $[0, 1]$ for which $f(x) - g(x)$ takes its maximum value, and explain why you chose this critical point for the answer. **Ans:** The derivative of $x - x^5$ is $1 - 5x^4$, and this is 0 when $x = 5^{-1/4}$. At the other critical points, the end points, the difference is 0.

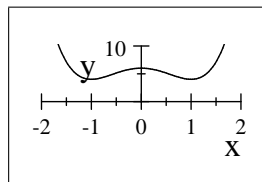
5 a) *Let $y = \ln x$ for $x > 0$. **Prove that** $dy/dx = 1/x$ **by applying implicit differentiation** to the equation $x = e^y$. **Show all your work.** **Ans:** Since $x = e^y$, $1 = e^y(dy/dx)$, or $dy/dx = 1/e^y = 1/x$.

- 5 b) *Suppose $0 \leq \theta \leq \pi/2$ and $\tan \theta = 5/3$. Draw and label the appropriate triangle and use it to find $\sin \theta$ and $\sec \theta$. **Ans:** For the appropriate triangle, the adjacent side is 3, the opposite side is 5, and the hypotenuse is $\sqrt{34}$. Therefore, $\sin \theta = 5/\sqrt{34}$ and $\sec \theta = \sqrt{34}/3$.
- 5 c) *For what special value of $a > 1$ is $\lim_{\Delta x \rightarrow 0} \frac{a^{\Delta x} - a^0}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{a^{\Delta x} - 1}{\Delta x} = 1$? **Ans:** e .
- 6) *A train 16 feet wide is approaching a calm observer standing in the middle of the track. The train is traveling at 32 ft/sec. Let y be the distance from the observer to the middle of the front of the train, and let z be the distance from the observer to the edge of the front of the train at the observer's right. Note that $y^2 + 8^2 = z^2$. Let θ be the angle in radians between the line going from the observer to the middle of the front of the train and the line going from the observer to the edge of the front of the train at the observer's right. Find the rate of increase (in radians per second) of the angle θ when $y = 24$ feet. (Hint: First express y as a constant times a trigonometric function of θ .) **Ans:** Since $y = 8 \cot \theta$, and the distance from the observer to the train is decreasing, $dy/dt = -32 = -8 \csc^2 \theta (d\theta/dt)$, or $d\theta/dt = 4 \sin^2 \theta$. At the moment in question, $y = 24$, so $\sin \theta = 1/\sqrt{10}$, so $d\theta/dt = 4/10 = 2/5$ radians/sec.

Sample Problem: Find the critical points and local maxima and minima for $f(x) = 2x^4 - 4x^2 + 6$ on the real line.

Ans: There are no end points, and no points where the derivative does not exist. The derivative $f'(x) = 8x^3 - 8x = 0$ when $x = 0$ and when $x^2 = 1$, i.e. $x = \pm 1$. To test these three points, we note that $f'(-2) < 0$, $f'(-1/2) > 0$, $f'(1/2) < 0$, and $f'(2) > 0$, so f has a local minimum at -1 with $f(-1) = 4$, f has a local maximum at 0 with $f(0) = 6$, and f has a local minimum at 1 with $f(1) = 4$. We can also check this using the second derivative $f''(x) = 24x^2 - 8$. Since $f''(-1) > 0$, and $f''(1) > 0$, -1 and 1 are local minima of f . Since $f''(0) < 0$, 0 is a local maximum of f . Notice that $f(x)$ goes to $+\infty$ as x goes to either $+\infty$ or $-\infty$. It follows that there is no absolute maxima value for the function, but the value 4 achieved at -1 and $+1$ is an absolute minimum value.

$$2x^4 - 4x^2 + 6$$



Sample Problem: Find $\lim_{x \rightarrow 0} \frac{2x^2}{x \sin x}$. **Ans:** Using L'Hôpital's Rule twice, we have

$$\lim_{x \rightarrow 0} \frac{2x^2}{x \sin x} = \lim_{x \rightarrow 0} \frac{4x}{\sin x + x \cos x} = \lim_{x \rightarrow 0} \frac{4}{\cos x + \cos x - x \sin x} = 2.$$

Sample Problem Find $\int \cos(x^2 + 4x + 1)(x + 2) dx$. **Ans:** Here, we let $u = x^2 + 4x + 1$, so that $du = (2x + 4) dx$, and $\frac{1}{2} du = (x + 2) dx$. In terms of u , we want $\frac{1}{2} \int \cos u du = \frac{1}{2} \sin u + C$. The answer, which must be in terms of x , is then $\frac{1}{2} \sin(x^2 + 4x + 1) + C$.

Sample Problem: A ball is tossed up from the edge of a 96 foot building with an initial velocity of 16 feet per second. Find the velocity and height of the ball at any time t , the time when the ball reaches its highest point, and the time when the ball hits the ground.

Answer: The acceleration at any time t is $a(t) = -32$ feet/sec², the velocity $v(t) = -32t + 16$ feet/sec, and the height $y(t) = -16t^2 + 16t + 96$. The maximum value of y is when $v(t) = 0$, that is at $1/2$ of a second. The ball hits when ground when $t^2 - t - 6 = 0$, that is, $t = -2$ and $t = 3$ seconds; only the second answer is correct since it must be positive.

Sample Problem: Evaluate the sum $\sum_{j=1}^3 j^4$. **Ans** $1^4 + 2^4 + 3^4 = 1 + 16 + 81 = 98$.

Sample Problem: Solve the initial value problem $\frac{dy}{dx} = (x^2 - 2x)(x - 1)$, $y(1) = 3$.

Ans: $\int (x^2 - 2x)(x - 1) dx$ can be evaluated by setting $u = x^2 - 2x$, so $du = (2x - 2) dx$, and $\frac{1}{2} du = (x - 1) dx$. With this substitution, the integral is $\frac{1}{2} \int u du = \frac{1}{4} u^2 + C = \frac{1}{4} (x^2 - 2x)^2 + C$. We want the value 3 when $x = 1$, so $C = 3 - \frac{1}{4}$.

Sample Problem. Evaluate $\int_2^5 e^{-t^2} dt + \int_5^2 e^{-x^2} dx$. **Ans:** 0.

Sample Problem. Evaluate $\int_2^3 2x dx$. **Ans.** Since $D_x x^2 = 2x$, the answer is $3^2 - 2^2 = 5$.

Sample Problem Let $f(x) = x^2$. Find the average value of f on the interval $[0, 3]$ and a point where f takes that average value. **Ans.**

$$y_{avg} = \frac{1}{3} \int_0^3 x^2 = \frac{1}{9} 3^3 = 3$$

is the average value of f on $[0, 3]$, and a point where f takes that average value is $c = \sqrt{3}$; that is, $y_{avg} = f(\sqrt{3})$.

Sample Problem. Write a formula for a function of x on the real line for which the derivative is e^{-x^2} and the function takes the value 0 at $x = 0$. **Ans.** $\int_0^x e^{-t^2} dt$.

EXAM 3.

1 a) Find $\int x^6 + 4x^3 - 3x + 2 dx$. **Ans.** $\frac{1}{7}x^7 + x^4 - \frac{3}{2}x^2 + 2x + C$.

b) Find $\int \sin(x^3 + 6x + 1)(x^2 + 2) dx$. **Ans:**

Let $u = x^3 + 6x + 1$, so $du = (3x^2 + 6) dx$, so $1/3 du = (x^2 + 2)dx$. Then

$$\begin{aligned} \int \sin(x^3 + 6x + 1)(x^2 + 2) dx &= \frac{1}{3} \int \sin u du = -\frac{1}{3} \cos u + C \\ &= -\frac{1}{3} \cos(x^3 + 6x + 1) + C. \end{aligned}$$

1 c) Solve the initial value problem $\frac{dy}{dx} = 1/x$, $y(e) = 3$. **Ans.** $y = \ln x + C$. Since $\ln e = 1$, $C = 2$, so $y = \ln x + 2$.

2) *Given a differentiable function $y = f(x)$, you want to find a value x such that $f(x) = 0$. You will use Newton's rule. You make a guess x_1 such that the derivative $f'(x_1) \neq 0$.

a) Write the formula for the tangent line L at the point $(x_1, f(x_1))$. **Ans:** $y - f(x_1) = f'(x_1)(x - x_1)$.

b) Your next guess x_2 is the x -coordinate of the point where that tangent line L intersects what line? **Ans.** The x -axis.

3 a) *Complete the following: For $y = x^{2/3}$, the differential $dy = \dots$ **Ans.** $dy = \frac{2}{3}x^{-1/3}dx$

3 b) *Use the answer to Part a and the fact that $27^{2/3} = 9$ to estimate $25^{2/3} - 9$. **Ans.**

$$dy = \frac{2}{3}(27)^{-1/3}(-2) = \frac{-4}{9} = -0.44444. \quad \text{Note: } 25^{2/3} - 9 = -0.45012$$

4a) $\lim_{x \rightarrow 0} \frac{3 \tan 5x}{2x} = ?$ **Ans:**

$$\lim_{x \rightarrow 0} \frac{3 \tan 5x}{2x} = \lim_{x \rightarrow 0} \frac{D_x(3 \tan 5x)}{D_x(2x)} = \lim_{x \rightarrow 0} \frac{15 \sec^2 5x}{2} = \frac{15}{2}.$$

4b) $\lim_{x \rightarrow +\infty} \frac{4x^2 - 2x + 8}{3x^2 - 3x + 4} = ?$ **Ans:** $4/3$.

- 5) *Let y be a function of x satisfying the equation $4 \sin y = x^3$. Implicitly differentiate this equation twice to obtain an equation involving x , y , dy/dx , and d^2y/dx^2 . Ans:

$$\begin{aligned} 4 \sin y &= x^3 \\ 4 \cos y \frac{dy}{dx} &= 3x^2 \\ -4 \sin y \left(\frac{dy}{dx} \right)^2 + 4 \cos y \frac{d^2y}{dx^2} &= 6x. \end{aligned}$$

- 6) A function f has a second derivative everywhere on the real line. In each of the following cases, what can you say about f and its graph over an open interval (a, b) if you are given the following information?

Case 1) The first derivative $f'(x) = 0$ for each x in (a, b) . The function is constant.

Case 2) The first derivative $f'(x) < 0$ for each x in (a, b) . The function is decreasing

Case 3) The second derivative $f''(x) > 0$ for each x in (a, b) . The graph is concave up.

Case 4) The second derivative $f''(x)$ changes sign at a point $x = c \in (a, b)$. The point c is an inflection point where the concavity changes.

- 7) *Suppose f is a differentiable function on the real line and the derivative $f'(x) \leq 2$ for all x . Is it possible that $f(1) = 4$ and $f(3) = 10$? **Explain.** Ans. No because

$$\frac{f(3) - f(1)}{3 - 1} = 3,$$

and $f'(x)$ is never 3, so by the Mean Value Theorem, this can never happen.

- 8) *A ball is tossed up from the edge of a 192 foot (that's $16 \cdot 12$ foot) building with an initial velocity of 16 feet per second. It goes up and then falls to ground level and continues falling into a narrow pipe that opens at ground level. A camera is mounted at ground level 16 feet from the opening of the pipe. Let ground level be height $y = 0$. Remember, the acceleration due to gravity is $a = -32$ feet/sec².

- a) Find a formula for the height y of the ball at any time t in seconds after the ball is tossed up. Ans.

$$\begin{aligned} a(t) &= -32 \\ v(t) &= -32t + 16 \\ y(t) &= -16t^2 + 16t + 192. \end{aligned}$$

- b) When will the ball reach its maximum height? Ans. When $v = 0$ at $t = 1/2$ second.
- c) When will the ball enter the pipe at ground level? Ans: $y(t) = 0$ when $t^2 - t - 12 = (t - 4)(t + 3) = 0$, and the positive time when this happens is at $t = 4$ seconds.
- d) Find how fast the angle θ between ground level and the line from the camera to the ball is changing when the ball enters the pipe, that is, when $\theta = 0$. Show all your work. Ans.

$$\begin{aligned} \tan \theta &= \frac{y(t)}{16} = -t^2 + t + 12 \\ \sec^2 \theta \frac{d\theta}{dt} &= -2t + 1 \end{aligned}$$

At $t = 4$, $\theta = 0$, $\sec^2 \theta = 1$, and $\frac{d\theta}{dt} = -7$ radians per second.

Sample Problem: Change the limits of integration and evaluate $\int_0^1 \cos(\pi x^2)(2\pi x)dx$.
 Ans: Let $u = \pi x^2$, $du = 2\pi x dx$, when $x = 0$, $u = 0$, when $x = 1$, $u = \pi$, so

$$\int_0^1 \cos(\pi x^2)(2\pi x)dx = \int_0^\pi \cos u du = [\sin u]_0^\pi = 0.$$

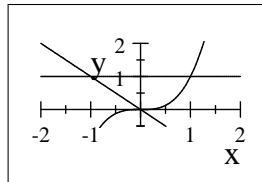
Note that here we did not go back to the old variable x .

Sample Problem: Solve $\int_2^3 x^2 \sqrt{1 + x^3} dx$. **Ans.** Using the substitution $u = 1 + x^3$, $\frac{1}{3} du = x^2 dx$, we note that when $x = 2$, $u = 9$, and when $x = 3$, $u = 28$. We then have

$$\int_2^3 x^2 \sqrt{1 + x^3} dx = \int_9^{28} \frac{1}{3} \sqrt{u} du = \left[\frac{2}{9} u^{\frac{3}{2}} \right]_9^{28} = \frac{2}{9} (28^{\frac{3}{2}} - 27).$$

Sample Problem: Find the region bounded by the curves $y = x^3$, $y = -x$, and $y = 1$ and find the area of the region first by integrating with respect to x and then integrating with respect to y .

x^3



Ans: As shown, the curves intersect in pairs at $(-1, 1)$, $(0, 0)$, and $(1, 1)$. If we integrate with respect to x , we need two integrals for the area:

$$\begin{aligned} \int_{x=-1}^{x=0} (1 - (-x)) dx + \int_{x=0}^{x=1} (1 - x^3) dx &= \left[x + \frac{1}{2}x^2 \right]_{-1}^0 + \left[x - \frac{1}{4}x^4 \right]_0^1 \\ &= 0 - \left(-1 + \frac{1}{2} \right) + 1 - \frac{1}{4} = \frac{5}{4}. \end{aligned}$$

If we integrate with respect to y , we only need one integral for the area. Here it is the function for the right most graph minus the other function:

$$\int_{y=0}^{y=1} (y^{1/3} - (-y)) dy = \left[\frac{3}{4}y^{4/3} + \frac{1}{2}y^2 \right]_0^1 = \frac{3+2}{4} = \frac{5}{4}.$$

Sample Problem: Evaluate $\int x(x+3)^7 dx$. **Ans:** For the integral $\int x(x+3)^7 dx$, we would like to simplify $x+3$. Therefore, we let $u = x+3$, so $x = u-3$, and $du = dx$. We then have

$$\int x(x+3)^7 dx = \int (u-3)u^7 du = \int u^8 - 3u^7 du.$$

Sample Problem. Suppose the mass density of a 10 centimeter rod with the left end at 0 is given by $\rho(x) = x^2$ grams/cm. Find the mass of the rod and the average density. **Ans.** The mass is $\int_0^{10} x^2 dx = \frac{1000}{3}$ grams. The average density is $\frac{1}{10} \int_0^{10} x^2 dx = \frac{100}{3}$ grams/cm.

Sample Problem. Suppose the velocity of a moving body at time t is given by $v(t) = 1 - t$ for $0 \leq t \leq 2$. Find the net distance traveled and the total distance traveled for that interval of time.

Ans. The net distance traveled is

$$\int_0^2 1 - t dt = \left[t - \frac{1}{2}t^2 \right]_0^2 = 0,$$

but the total distance traveled is

$$\begin{aligned} \int_0^2 |1 - t| dt &= \int_0^1 1 - t dt + \int_1^2 t - 1 dt \\ &= \left[t - \frac{1}{2}t^2 \right]_0^1 + \left[\frac{1}{2}t^2 - t \right]_1^2 \\ &= \frac{1}{2} + \frac{1}{2} = 1. \end{aligned}$$

Sample Problem. The base of a certain solid is a circle of radius a and center 0 in the x - y -plane. Find the volume of the solid if each cross section perpendicular to the x -axis is a square.

Ans. For each x between $-a$ and a , the base of the cross section has length $2\sqrt{a^2 - x^2}$, so $A(x) = 4(a^2 - x^2)$. This is an even function of x . It follows that the volume of the solid is

$$\int_{-a}^a 4(a^2 - x^2) dx = 2 \int_0^a 4(a^2 - x^2) dx = 8(a^2x - \frac{1}{3}x^3)_0^a = 8(a^3 - \frac{1}{3}a^3) = \frac{16}{3}a^3.$$

Sample Problem. Find the volumes of the two solids you get by rotating the area between the line $y = x$ and $y = x^2$ about the x -axis and about the y -axis. **Ans.** The volume of the first solid is

$$\pi \int_0^1 [x^2 - x^4] dx = \pi \left[\frac{1}{3}x^3 - \frac{1}{5}x^5 \right]_0^1 = \pi \frac{2}{15}.$$

For the second solid, we want to rotate the area between the line $x = y$ and the curve $x = \sqrt{y}$ about the y -axis. The volume of the corresponding solid is

$$\pi \int_0^1 [y - y^2] dy = \pi \left[\frac{1}{2}y^2 - \frac{1}{3}y^3 \right]_0^1 = \pi \frac{1}{6}.$$

Sample Problem. We want to rotate the area between the line $y = x$ and $y = x^2$ about the line $y = -2$. Find the volume of the corresponding solid. **Ans.**

$$\begin{aligned} \pi \int_0^1 [(x+2)^2 - (x^2+2)^2] dx &= \pi \int_0^1 -x^4 - 3x^2 + 4x dx \\ &= \pi \left(-\frac{1}{5} - 1 + 2 \right) = \frac{4\pi}{5}. \end{aligned}$$

Sample Problem. Write the integral for the total work necessary to empty a half spherical tank of water of radius r feet resting on the ground, where the water must be moved to the point $y = H \geq r$. The density of water is 62.4 pounds per cubic foot. **Ans:** First, we note that each cross section below the level $y = r$ has radius

$$s = \sqrt{r^2 - (r - y)^2} = \sqrt{2yr - y^2}.$$

The total work done is given by the integral

$$62.4\pi \int_0^r (H - y)(2yr - y^2) dy \text{ foot-pounds}$$

Sample Problem: A rope is 80 feet long and weighs $\frac{3}{10}$ of a pound per foot. It has one end at the top of a tall building and the other end hanging down 80 feet from the top of the building. The rope is to be pulled onto the top of the building. Calculate the work.

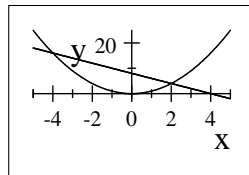
Ans: If we let y be the distance to the bottom end of the rope, then the section from y to $y + dy$ weighs $\frac{3}{10} dy$ of a pound and must be moved $80 - y$ feet, so the work is

$$\int_0^{80} \frac{3}{10}(80 - y)dy = 960 \text{ foot-pounds.}$$

If we let x be the length of rope that has been pulled over the side of the building, then in lifting from x to $x + dx$, the weight is $\frac{3}{10}(80 - x)$, and it is moved dx , so we get the same answer.

Sample Problem: For the region between the curves $y = 8 - 2x$ and the curve $y = x^2$ from 0 to 2, find the area, the moments about the y and x axes, and the centroid.

x^2



Ans: the area is

$$A = \int_0^2 dA = \int_0^2 (8 - 2x - x^2) dx = \frac{28}{3}.$$

The moment about the y -axis is

$$M_y = \int_0^2 x dA = \int_0^2 (x \cdot (8 - 2x - x^2)) dx = \frac{20}{3}.$$

The moment about the x -axis is

$$M_x = \frac{1}{2} \int_0^2 ((8 - 2x)^2 - (x^2)^2) dx = \frac{512}{15}.$$

The centroid is given by

$$\bar{x} = \frac{M_y}{A} = \frac{\frac{20}{3}}{\frac{28}{3}} = \frac{5}{7} = 0.7143$$

$$\bar{y} = \frac{M_x}{A} = \frac{\frac{512}{15}}{\frac{28}{3}} = \frac{128}{35} = 3.6571.$$

Sample Problem: Evaluate $D_x \log_{10} x$. **Ans:** $D_x \frac{\ln x}{\ln 10} = \frac{1}{x \cdot \ln 10}$.

Sample Problem: Evaluate $\int \frac{dx}{x \cdot \log_2 x}$. **Ans:**

$$\int \frac{dx}{x \cdot \log_2 x} = \int \frac{dx}{x \cdot \frac{\ln x}{\ln 2}} = \ln 2 \int \frac{dx}{x \cdot \ln x} = \ln 2 \cdot \ln |\ln x| + C.$$

Sample Problem: Find the derivative $D_x 2^x$. **Ans:**

$$D_x 2^x = D_x e^{x \ln 2} = \ln 2 \cdot e^{x \ln 2} = \ln 2 \cdot 2^x.$$

Sample Problem: Evaluate $D_x \sinh(e^x)$. **Ans.** $D_x \sinh(e^x) = e^x \cdot \cosh(e^x)$.

EXAM 4:

1 a) Assume f is an odd continuous function; that is, for each $x \geq 0$, $f(-x) = -f(x)$. Given that $\int_0^7 f(x) dx = 3$, what is the value of the integral $\int_{-7}^0 f(x) dx$? **Ans:** -3 .

1 b) Assume g is an even continuous function; that is, for all $x \geq 0$, $g(-x) = g(x)$. Given that $\int_0^7 g(x) dx = 3$, what is the value of the integral $\int_{-7}^0 g(x) dx$? **Ans:** $+3$.

1 c) Evaluate $\int_2^5 e^{3t^2} dt + \int_5^2 e^{3x^2} dx$. **Ans:** 0 .

1d) What is the average value of the function $\sec^2 x$ on the interval $[0, \frac{\pi}{4}]$? **Ans:**

$$\frac{1}{\pi/4} \int_0^{\pi/4} \sec^2 x dx = \frac{4}{\pi} \tan \frac{\pi}{4} = \frac{4}{\pi}.$$

2 a) Evaluate $\sum_{i=1}^3 i^5$. **Ans:** $\sum_{i=1}^3 i^5 = 1^5 + 2^5 + 3^5 = 1 + 32 + 243 = 276$.

2 b) Evaluate $\int (\sin 3x^2)^5 \cdot \cos 3x^2 \cdot 6x dx$.

Ans: Let $u = \sin 3x^2$, so $du = \cos 3x^2 \cdot 6x dx$. Then

$$\int (\sin 3x^2)^5 \cdot \cos 3x^2 \cdot 6x dx = \int u^5 du = \frac{1}{6} u^6 + C = \frac{1}{6} (\sin 3x^2)^6 + C.$$

- 2 c) *Change the variable **and** the limits of integration by a substitution, and then evaluate: $\int_0^1 \frac{1}{(x^4+1)^2} \cdot x^3 dx$. **Ans:** Let $u = x^4 + 1$. Then $du = 4x^3 dx$, or $\frac{1}{4} du = x^3 dx$. Now

$$\int_0^1 \frac{1}{(x^4 + 1)^2} \cdot x^3 dx = \frac{1}{4} \int_1^2 u^{-2} du = \frac{1}{4} \left[\frac{-1}{u} \right]_1^2 = \frac{1}{8}.$$

- 3) *Let $f(x) = \cos x$ on the interval $[0, \pi/2]$. Write out, **but do not add up**, the following sums for $\Delta x = \pi/6$. (Hint: There are three intervals. If you forget how to evaluate $\cos x$, remember the side opposite the $\pi/6$ angle of the appropriate right triangle has half the length of the hypotenuse.)

Table: $\cos 0 = 1$, $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$, $\cos \frac{\pi}{3} = \frac{1}{2}$, $\cos \frac{\pi}{2} = 0$.

- a) The Riemann sum evaluating at the left of each interval. Since f is decreasing, this is also the upper sum $\bar{A}_f(\Delta x)$. **Ans:** $\left(1 + \frac{\sqrt{3}}{2} + \frac{1}{2}\right) \cdot \frac{\pi}{6}$
- b) The lower sum $\underline{A}_f(\Delta x)$. **Ans:** $\left(\frac{\sqrt{3}}{2} + \frac{1}{2} + 0\right) \cdot \frac{\pi}{6}$
- c) The trapezoidal approximation using $\Delta x = \pi/6$. Use the method discussed in the lecture. **Ans:** $\left(\frac{1}{2} + \frac{\sqrt{3}}{2} + \frac{1}{2} + \frac{0}{2}\right) \cdot \frac{\pi}{6}$
- d) Given a continuous function $f(x)$ on an interval $[a, b]$ and $\Delta x > 0$, we have written M_i for the maximum value of $f(x)$ in the i^{th} interval and m_i for the minimum value of $f(x)$ in the i^{th} interval. We then set $E_f(\Delta x) = \max_i(M_i - m_i)$. State the basic fact about $E_f(\Delta x)$ that is needed for integration. **Ans:** $\lim_{\Delta x \rightarrow 0} E_f(\Delta x) = 0$.

- 4 a) Let $F(x) = \int_2^x e^{t^2+3} dt$. What is the derivative $F'(x)$? **Ans:** e^{x^2+3} .

- 4 b) What is the value $F(2)$? **Ans:** 0.

- 4 c) *Write a formula for the antiderivative of e^{x^2+3} that takes the value 0 at $x = 2$. (Hint: look at Parts a and b.) **Ans:** $\int_2^x e^{t^2+3} dt$.

- 5) Suppose the mass density of a 3 centimeter long rod with the left end at 0 is given by $\rho(x) = x^3$ grams/cm. Find the mass of the rod. **Ans:** $\int_0^3 x^3 dx = \left[\frac{1}{4}x^4\right]_0^3 = \frac{81}{4}$.

- 6) *Let A be the area of the region to the right of the y -axis bounded by the curves $y = x^3$ and $y = \sqrt{x}$. Note that for $0 \leq x \leq 1$, $x^3 \leq \sqrt{x}$.

a) Find the area A integrating with respect to x . Show all your work. **Ans:**

$$\int_0^1 (\sqrt{x} - x^3) dx = \left[\frac{2}{3}x^{3/2} - \frac{1}{4}x^4 \right]_0^1 = \frac{2}{3} - \frac{1}{4} = \frac{8-3}{12} = \frac{5}{12}.$$

b) Find the area A integrating with respect to y . Show all your work. **Ans:**

$$\int_0^1 (y^{1/3} - y^2) dy = \left[\frac{3}{4}y^{4/3} - \frac{1}{3}y^3 \right]_0^1 = \frac{3}{4} - \frac{1}{3} = \frac{9-4}{12} = \frac{5}{12}.$$

7) *Let R be the region to the right of the y -axis bounded by the x -axis, the curve $y = x^3$, and the vertical line $x = 1$.

a) Find the volume obtained by rotating the region R about the x -axis.

Ans: $\pi \int_0^1 x^6 dx = \pi \left[\frac{1}{7}x^7 \right]_0^1 = \frac{\pi}{7}.$

b) The base of a certain solid is the region R . Find the volume of the solid if each cross section perpendicular to the x -axis is a square. **Ans:** $\int_0^1 x^6 dx = \left[\frac{1}{7}x^7 \right]_0^1 = \frac{1}{7}.$

Sample Problem: Evaluate $\int_0^1 \frac{dx}{\sqrt{4-x^2}}$ in two ways. **Ans:** Let $x = 2 \sin \theta$, so $dx = 2 \cos \theta d\theta$. Then

$$\int_0^1 \frac{dx}{\sqrt{4-x^2}} = \int_0^{\pi/6} \frac{2 \cos \theta d\theta}{2 \cos \theta} = \frac{\pi}{6}.$$

Using arcsin,

$$\int_0^1 \frac{dx}{\sqrt{4-x^2}} = \frac{1}{2} \int_0^1 \frac{dx}{\sqrt{1-\left(\frac{x}{2}\right)^2}} = \left[\arcsin \frac{x}{2} \right]_0^1 = \frac{\pi}{6}.$$

Sample Problem For $a = 2$, evaluate $\int_0^{2\sqrt{3}} \frac{dx}{4+x^2}$ in two ways. **Ans:** With the above substitution, when $x = 0$, $\theta = 0$, and when $x = 2\sqrt{3}$, $\theta = \pi/3$. We therefore have

$$\int_0^{2\sqrt{3}} \frac{dx}{4+x^2} = \int_0^{\pi/3} \frac{2 \sec^2 \theta d\theta}{4 \sec^2 \theta} = \frac{1}{2} \int_0^{\pi/3} d\theta = \frac{\pi}{6}.$$

We can also see that

$$\int_0^{2\sqrt{3}} \frac{dx}{4+x^2} = \frac{1}{4} \int_0^{2\sqrt{3}} \frac{dx}{1+\left(\frac{x}{2}\right)^2} = \frac{1}{2} \arctan \left(\frac{x}{2} \right) \Big|_0^{2\sqrt{3}} = \frac{\pi}{6}.$$

Sample Problem: We have seen that if \$100 is put in the bank at 5% and left for 20 years, the amount of money in the account at the end of 20 years is $\$100 \cdot (1 + \frac{5}{100})^{20} = \265.33 . How much more would be in the account at the end of 20 years if the money had compounded continuously? **Ans.** In this case, there would be $\$100e^{\frac{5}{100} \cdot 20} = \$100e = \$271.83$, that is \$6.50 more.

Sample Problem: Suppose a radioactive material decays at a rate given by a negative constant $-k$. That is the amount of material at time t is $Q(t) = Q_0e^{-kt}$. What is the half life of this material?

Ans: Here we want the value of t such that $Q(t) = \frac{1}{2}Q_0$, that is, $e^{-kt} = \frac{1}{2}$. Taking natural logarithms of both sides of this equation we have $-kt = -\ln 2$, or $t = \frac{\ln 2}{k}$.

Sample Problem: Here is the problem using the arctan function. A rectangular painting is hung on a wall. We want to know how high the painting should be so that the top and bottom subtend the maximum angle for the viewer. We may assume that the viewer is a feet from the picture, and the picture is b feet high. Note that a and b are constants. We will also assume that the viewer's eyes are at level $y = 0$, and that the bottom of the painting is at y , which may be positive or negative or 0.

Ans. The angle subtended by the picture is then $A(y) = \arctan(\frac{b+y}{a}) - \arctan(\frac{y}{a})$ for all possible values of y . It is clear that as y tends to $+\infty$, or $-\infty$, $A(y)$ tends to 0. If there is only one critical point of $A(y)$, on the real line, that must be where the maximum value of $A(y)$ occurs. Now

$$A'(y) = \frac{1}{1 + (\frac{b+y}{a})^2} \cdot \frac{a}{a^2} - \frac{1}{1 + (\frac{y}{a})^2} \cdot \frac{a}{a^2} = \frac{a(a^2 + y^2) - a(a^2 + (b+y)^2)}{(a^2 + (b+y)^2)(a^2 + y^2)}.$$

This is 0 when $(a^2 + y^2) - (a^2 + (b+y)^2) = 0$. That is, $A'(y) = 0$ when $2by + b^2 = 0$, i.e., when $y = -b/2$. It follows that, the middle of the picture should be at eye level.