Math 241 Section DD6 Quiz 1: *Solutions*

1. (10 pts) Given $\mathbf{a} = \langle 5, 2, 9\rangle$ and $\mathbf{b} = \langle 1, 2, 1\rangle$, find $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{a} = \mathbf{u} + \mathbf{v}$, $\mathbf{u} \| \mathbf{b}$ and $\mathbf{v} \perp \mathbf{b}$.

Let

$$\mathbf{u} = \text{proj}_{\mathbf{b}}\,\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b} = \Big(\frac{5\cdot 1 + 2\cdot 2 + 9\cdot 1}{1^2 + 2^2 + 1^2}\Big)\langle 1, 2, 1\rangle = \frac{5+4+9}{6}\langle 1, 2, 1\rangle \quad (1)$$

$$= 3\langle 1, 2, 1\rangle = \langle 3, 6, 3\rangle \quad (2)$$

This vector is clearly parallel to $\mathbf{b}$ since $\mathbf{u} = 3\mathbf{b}$. Now let us find $\mathbf{v}$ by taking

$$\mathbf{v} = \mathbf{a} - \mathbf{u} = \langle 5, 2, 9\rangle - \langle 3, 6, 3\rangle = \langle 2, -4, 6\rangle \quad (3)$$

This works since $\mathbf{v}\cdot\mathbf{b} = 2 - 8 + 6 = 0$ so it is perpendicular to $\mathbf{b}$.

2. (10 pts with 2 parts) Given the points $P = (2, 0, -3)$, $Q = (3, 1, 0)$, and $R = (5, 2, 2)$

2a) Find the area of the triangle $PQR$

We know that $\overrightarrow{PQ} = \langle 3-2, 1-0, 0-(-3)\rangle = \langle 1, 1, 3\rangle$ and $\overrightarrow{PR} = \langle 5-2, 2-0, 2-(-3)\rangle = \langle 3, 2, 5\rangle$. Since the triangle $PQR$ is contained in the parallelogram made by $\overrightarrow{PR}$ and $\overrightarrow{PQ}$; and in addition the line from $\overleftrightarrow{QR}$ divides the parallelogram into two congruent triangles, finding the area of the parallelogram by the cross product formula and taking $\frac{1}{2}$ of it is the right way to go.

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 3 \\ 3 & 2 & 5 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 3 \\ 1 & 0 & -1 \end{vmatrix} \quad (4)$$

The last step is because the determinant of a matrix $A$ and the determinant of a matrix $C$ formed by adding a constant multiple of a row of $A$ to another row of $A$ are equal. So taking $-2$ times the second row and adding it to the third row does not change the determinant.

Expanding (4) it becomes

$$-\mathbf{j}\begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{k} \\ 1 & -1 \end{vmatrix} = -\mathbf{i} + 4\mathbf{j} - \mathbf{k} \tag{5}$$

Then

$$\|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \sqrt{18} = 3\sqrt{2} \tag{6}$$

and the Area of $PQR$ is $\frac{3\sqrt{2}}{2}$.

2b) Find two unit vectors orthogonal to the plane through $PQR$.

Taking $\pm\frac{\overrightarrow{PQ}\times\overrightarrow{PR}}{\|\overrightarrow{PQ}\times\overrightarrow{PR}\|}$ will give the two unit vectors we are after. They are equal to $\pm\frac{1}{3\sqrt{2}}\langle -1, 4, -1\rangle$ by using the last part of (5) to get $\overrightarrow{PQ} \times \overrightarrow{PR}$ and (6) to get $\|\overrightarrow{PQ} \times \overrightarrow{PR}\|$.