In exercises 1,3,and 5 find the equation of the tangent plane and the normal line to the surface at the given point.

1. $f(x, y) = z = x^2 + y^2 - 1$ at $(2, 1, 4)$ and $(2, 0, 3)$.

We know that $f_x = 2x$ and $f_y = 2y$ so the tangent plane is

$$z - 4 = 4(x - 2) + 2(y - 1) \tag{1}$$

and the normal line is

$$x = 2 + 4t \tag{2}$$

$$y = 1 + 2t \tag{3}$$

$$z = 4 - t \tag{4}$$

since we take the slope vector $\langle 4, 2, -1\rangle$ and make a line that goes through $(2, 1, 4)$.

For the point $(2, 0, 3)$ the tangent plane is

$$z - 3 = 4(x - 2) \tag{5}$$

and the normal line is

$$x = 2 + 4t \tag{6}$$

$$y = 0 \tag{7}$$

$$z = 3 - t \tag{8}$$

3. $f(x, y) = z = \sin(x)\cos(y)$ at $(0, \pi, 0)$ and $(\frac{\pi}{2}, \pi, -1)$.

We know that $f_x = \cos(x)\cos(y)$ and $f_y = -\sin(x)\sin(y)$ so the tangent plane is

$$z = -(x - \pi) \tag{9}$$

and the normal line is

$$x = -t \tag{10}$$

$$y = \pi \tag{11}$$

$$z = -t \tag{12}$$

since we take the slope vector $\langle -1, 0, -1 \rangle$ and make a line that goes through $(0, \pi, 0)$.

For the point $(\frac{\pi}{2}, \pi, -1)$ the tangent plane is

$$z = -1 \tag{13}$$

and the normal line is

$$x = \frac{\pi}{2} \tag{14}$$

$$y = \pi \tag{15}$$

$$z = -t \tag{16}$$

5. $f(x, y) = z = \sqrt{x^2 + y^2}$ at $(-3, 4, 5)$ and $(8, -6, 10)$.

We know that

$$f_x = \frac{1}{2} 2x(x^2 + y^2)^{-\frac{1}{2}} = \frac{x}{\sqrt{x^2 + y^2}} \tag{17}$$

and

$$f_y = \frac{1}{2} 2y(x^2 + y^2)^{-\frac{1}{2}} = \frac{y}{\sqrt{x^2 + y^2}} \tag{18}$$

so the tangent plane is

$$5(z-5) = -3(x+3) + 4(y-4) \tag{19}$$

and the normal line is

$$x = -3 - \frac{3}{5}t \tag{20}$$

$$y = 4 + \frac{4}{5}t \tag{21}$$

$$z = 5 - t \tag{22}$$

since we take the slope vector $\langle -\frac{3}{5}, \frac{4}{5}, -1 \rangle$ and make a line that goes through $(-3, 4, 5)$.

For the point $(8, -6, 10)$ the tangent plane is

$$5(z-10) = 4(x-8) - 3(y+6) \tag{23}$$

and the normal line is

$$x = 8 + \frac{4}{5}t \tag{24}$$

$$y = -6 - \frac{3}{5}t \tag{25}$$

$$z = 10 - t \tag{26}$$

In exercises 7, 9, 11 compute the linear approximation of the function at the point.

7. $f(x,y) = \sqrt{x^2+y^2}$ at $(3,0)$ and $(0,-3)$. We know from the previous part of the problem that

$$f_x = \frac{1}{2}2x(x^2+y^2)^{-\frac{1}{2}} = \frac{x}{\sqrt{x^2+y^2}} \tag{27}$$

and

$$f_y = \frac{1}{2} 2y(x^2 + y^2)^{-\frac{1}{2}} = \frac{y}{\sqrt{x^2 + y^2}} \tag{28}$$

so

$$L(x, y) = 3 + (x - 3) \tag{29}$$

at $(3, 0)$. And at $(0, -3)$ we know that

$$L(x, y) = 3 + (y - 3) \tag{30}$$

9. $f(x, y) = xe^{xy^2} + 3y^2$ at $(0, 1)$ and $(2, 0)$. We know that

$$f_x = e^{xy^2} + xy^2 e^{xy^2} \tag{31}$$

and

$$f_y = 2yxe^{xy^2} + 6y \tag{32}$$

so

$$L(x, y) = 3 + x + 6(y - 1) \tag{33}$$

at $(0, 1)$. And at $(2, 0)$ we know that

$$L(x, y) = 2 + (x - 2) \tag{34}$$

11. $f(w, x, y, z) = w^2 xy - e^{wyz}$ at $(-2, 3, 1, 0)$ and $(0, 1, -1, 2)$. We know that

$$f_w = 2wxy - yze^{wyz} \tag{35}$$

$$f_x = w^2 y \tag{36}$$

$$f_y = w^2 x - wze^{wyz} \tag{37}$$

$$f_z = -wye^{wyz} \tag{38}$$

so

$$L(w,x,y,z) = 12 - 12(w+2) + 4(x-3) + 12(y-1) + 6z \tag{39}$$

at $(-2,3,1,0)$. And at $(0,1,-1,2)$ we know that

$$L(x,y) = -1 + 2w \tag{40}$$

In the exercises 23 and 25 find the increment Δz and write it in the form given in definition 4.1.

23. $f(x,y) = z = 2xy + y^2$

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y) = 2(x+\Delta x)(y+\Delta y) + (y+\Delta y)^2 - 2xy - y^2 \tag{41}$$

After expanding (41) we find that

$$\Delta z = (2x+2y)\Delta y + 2y\Delta x + (2\Delta x + 2\Delta y)(\Delta y) = f_y\Delta y + f_x\Delta x + \epsilon_1\Delta y \tag{42}$$

where $\epsilon_1 = 2\Delta x + 2\Delta y$

25. $f(x,y) = z = x^2 + y^2$

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y) = (x+\Delta x)^2 + (y+\Delta y)^2 - x^2 - y^2 \tag{43}$$

Then expanding (43) we arrive at

$$\Delta z = 2x\Delta x + 2y\Delta y + (\Delta x)^2 + (\Delta y)^2 \tag{44}$$

Then here if we let $\epsilon_1 = \Delta x$ and $\epsilon_2 = \Delta y$ then (44) equals

$$\Delta z = f_x \Delta x + f_y \Delta y + \epsilon_1(\Delta x) + \epsilon_2(\Delta y) \tag{45}$$

27. Determine whether or not $f(x,y) = z = x^2 + 3xy$ is differentiable.

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y) = (x+\Delta x)^2 + 3(x+\Delta x)(y+\Delta y) - x^2 - 3xy \tag{46}$$

After expanding eqref27.1 we arrive at

$$\Delta z = (2x+3y)\Delta x + 3x\Delta y + (\Delta y + \Delta x)\Delta x \tag{47}$$

We know that after labeling $\epsilon_1 = \Delta y + \Delta x$ we know that

$$\Delta z = f_x \Delta x + f_y \Delta y + \epsilon_1 \Delta x \tag{48}$$

Hence it is differentiable.

In exercise 29 find the total differential of $f(x,y)$

29. $f(x,y) = z = ye^x + \sin(x)$. We know that $f_x = ye^x + \cos(x)$ and that $f_y = e^x$ so we know that $dz = \big(ye^x + \cos(x)\big)dx + e^x dy$.

In exercise 31 show that the partial derivatives $f_x(0,0)$ and $f_y(0,0)$ both exist, but the function $f(x,y)$ is not differentiable at $(0,0)$.

31. Let

$$f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & \text{if}(x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases} \tag{49}$$

observe that

$$f_x = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = 0 \tag{50}$$

and

$$f_y = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = 0 \tag{51}$$

however if we try to take $\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$ at $y = x$ then this equals $\lim_{y\to 0} \frac{2y^2}{2y^2} = 1$. Thus the function $f(x,y)$ is not continuous at $(0,0)$, hence it is not differentiable at $(0,0)$.

43. Show that

$$f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2} & \text{if}(x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases} \tag{52}$$

It is continuous since for all points $(x,y) \neq (0,0)$ the function $f(x,y)$ is clearly continuous, and at $(0,0)$ we have that

$$\lim_{(x,y)\to(0,0)} |f(x,y)| \leq \lim_{y\to 0} |y| = 0 \tag{53}$$

thus it is continuous. However, at $f_x(0,0) = f_y(0,0) = 0$. This means that if $f(x,y)$ is differentiable that $f(x,y) = 0 + \epsilon_1 x + \epsilon_2 y$, but along $x = y$ we know that $f(x,x) = \frac{1}{2}x = 0 + \epsilon_1 x + \epsilon_2 x$ where $\epsilon_1(x,x)$ goes to 0 as $x \to 0$ and similarly with $\epsilon_2(x,x)$. This is a contradiction since $\frac{1}{2}x$ is linear while $\epsilon_1 x + \epsilon_2 x$ is not linear.