

Here for the test we have that there are some formulas that you want to be able to remember how to derive them and some formulas that you would like to memorize.

Memorize

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) \quad (1)$$

and

$$\|\mathbf{v}(t)\| = \frac{ds}{dt}(t) \quad (2)$$

and

$$\frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \mathbf{T}(t) \quad (3)$$

$$\frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \mathbf{N}(t) \quad (4)$$

$$\kappa(t) = \left| \frac{d\mathbf{T}}{ds}(t) \right| \quad (5)$$

Derivations Now knowing very little let us derive all meaningful formulas.

1. First meaningful formula.

Taking (3) and multiplying both sides by  $\|\mathbf{v}(t)\|$  we get

$$\mathbf{v}(t) = \|\mathbf{v}(t)\|\mathbf{T}(t) \quad (6)$$

then taking (6) and using (2) we see that

$$\mathbf{v}(t) = \frac{ds}{dt}(t)\mathbf{T}(t) \quad (7)$$

Then taking the derivative of (7) and using the power rule we find that this is

equal to

$$\mathbf{a}(t) = \mathbf{v}'(t) = \frac{d^2s}{dt^2}(t)\mathbf{T}(t) + \frac{ds}{dt}(t)\mathbf{T}'(t) \quad (8)$$

But then using the chain rule we have that  $\frac{d\mathbf{T}}{dt}(t) = \frac{d\mathbf{T}}{ds}(t)\frac{ds}{dt}(t)$  and by rule (5) we have that taking the magnitude and plugging into (8) we now have the desired formula

$$\mathbf{a}(t) = \frac{d^2s}{dt^2}(t)\mathbf{T}(t) + \left(\frac{ds}{dt}(t)\right)^2\kappa(t)\mathbf{N}(t) \quad (9)$$

(9) is the derivation we are looking for.

## 2. Second meaningful formula.

Because we wish to find the curvature we decide that it would be fun to look at

$$\|\mathbf{T}(t) \times \mathbf{a}(t)\| \quad (10)$$

Once we derived (9) we have that plugging (9) into (10) gives us that

$$\|\mathbf{T}(t) \times \left(\frac{d^2s}{dt^2}(t)\mathbf{T}(t) + \left(\frac{ds}{dt}(t)\right)^2\kappa(t)\mathbf{N}(t)\right)\| = \left(\left(\frac{ds}{dt}(t)\right)^2\kappa(t)\right)\|\mathbf{T}(t) \times \mathbf{N}(t)\| \quad (11)$$

Then (11) equals

$$\left(\left(\frac{ds}{dt}(t)\right)^2\kappa(t)\right)\|\mathbf{B}(t)\| = \left(\frac{ds}{dt}(t)\right)^2\kappa(t) = \|\mathbf{T}(t) \times \mathbf{a}(t)\| \quad (12)$$

Then we find that

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \quad (13)$$

and

$$\mathbf{a}(t) = \mathbf{r}''(t) \quad (14)$$

and

$$\frac{ds}{dt}(t) = \|\mathbf{r}'(t)\| \quad (15)$$

combine with (12) to give

$$\frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} \quad (16)$$