

Homework 10.3 Solns.

In exercises 1,3,and 5 compute  $\mathbf{a} \cdot \mathbf{b}$

$$\mathbf{a} = \langle 3, 1 \rangle \quad \mathbf{b} = \langle 2, 4 \rangle$$

$$\mathbf{a} \cdot \mathbf{b} = 3 \cdot 2 + 1 \cdot 4 = 10 \tag{1}$$

3.

$$\mathbf{a} = \langle 2, -1, 3 \rangle \quad \mathbf{b} = \langle 0, 2, 4 \rangle$$

$$\mathbf{a} \cdot \mathbf{b} = 0 + (-1) \cdot 2 + 3 \cdot 4 = -2 + 12 = 10 \tag{2}$$

5.

$$\mathbf{a} = \langle 2, 0, -1 \rangle \quad \mathbf{b} = \langle 0, 4, -1 \rangle$$

$$\mathbf{a} \cdot \mathbf{b} = 0 + 0 + (-1) \cdot (-1) = 1 \tag{3}$$

In exercise 7 and 9 compute the angle between the vectors

7.

$$\mathbf{a} = \langle 3, -2, 0 \rangle \quad \mathbf{b} = \langle 1, 1, 0 \rangle$$

$$\mathbf{a} \cdot \mathbf{b} = 3 - 2 = 1 = \|\mathbf{a}\| \|\mathbf{b}\| \cos(\theta) \tag{4}$$

However we know that  $\|\mathbf{a}\| = \sqrt{4 + 9} = \sqrt{13}$  and  $\|\mathbf{b}\| = \sqrt{2}$  so

$$\cos(\theta) = \frac{1}{\sqrt{26}} \tag{5}$$

and

$$\theta = \arccos\left(\frac{1}{\sqrt{26}}\right) \tag{6}$$

9.

$$\mathbf{a} = \langle 3, 1, -4 \rangle \quad \mathbf{b} = \langle -2, 2, 1 \rangle$$

$$\mathbf{a} \cdot \mathbf{b} = -6 + 2 - 4 = -8 \quad (7)$$

But this also equals

$$\|\mathbf{a}\| \|\mathbf{b}\| \cos(\theta) = \sqrt{4 + 4 + 1} \sqrt{9 + 1 + 16} \cos(\theta) = 3\sqrt{26} \cos(\theta) \quad (8)$$

So

$$\theta = \arccos\left(\frac{-8}{3\sqrt{26}}\right) \quad (9)$$

In 11 and 13 determine whether the vectors are orthogonal

11.

$$\mathbf{a} = \langle 2, -1 \rangle \quad \mathbf{b} = \langle 2, 4 \rangle$$

$$\mathbf{a} \cdot \mathbf{b} = 2 \cdot 2 - 4 = 0 \quad (10)$$

Yes they are orthogonal

13.

$$\mathbf{a} = \langle 6, 2, 0 \rangle \quad \mathbf{b} = \langle -1, 3, 0 \rangle$$

$$\mathbf{a} \cdot \mathbf{b} = -6 + 6 = 0 \quad (11)$$

These two are orthogonal as well.

In 15 and 17 find a vector perpendicular to the given vector.

15.

$$\mathbf{a} = \langle 2, -1 \rangle$$

Take a vector  $\mathbf{b} = \langle x, y \rangle$  then  $\mathbf{a} \cdot \mathbf{b} = 2x - y$  and this must equal 0 for  $\mathbf{b}$  to be perpendicular to  $\mathbf{a}$ . Take  $\mathbf{b} = \langle 3, 6 \rangle$ . In reality taking any point  $P$  on the line  $2x = y$  and looking at the position vector to  $P$  will give an orthogonal vector to  $\mathbf{a}$ .

17.

$$\mathbf{a} = \langle 6, 2, -1 \rangle$$

Take  $\mathbf{b} = \langle x, y, z \rangle$  then  $\mathbf{a} \cdot \mathbf{b} = 6x + 2y - z$  looking at any point  $P$  on the plane  $6x + 2y = z$  and its position vector will give an orthogonal vector to  $\mathbf{a}$ .

in 19 and 21 find  $\text{comp}_{\mathbf{b}} \mathbf{a}$  and  $\text{proj}_{\mathbf{b}} \mathbf{a}$ . 19.

$$\mathbf{a} = \langle 2, 1 \rangle \quad \mathbf{b} = \langle 3, 4 \rangle$$

$$\text{comp}_{\mathbf{b}} \mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|} = \frac{6 + 4}{5} = 2 \quad (12)$$

and

$$\text{proj}_{\mathbf{b}} \mathbf{a} = (\text{comp}_{\mathbf{b}} \mathbf{a}) \frac{1}{\|\mathbf{b}\|} \mathbf{b} = \frac{2}{5} \langle 3, 4 \rangle \quad (13)$$

21.

$$\mathbf{a} = \langle 2, -1, 3 \rangle \quad \mathbf{b} = \langle 1, 2, 2 \rangle$$

$$\text{comp}_{\mathbf{b}} \mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|} = \frac{6 + 2 - 2}{3} = 2 \quad (14)$$

and

$$\text{proj}_{\mathbf{b}} \mathbf{a} = (\text{comp}_{\mathbf{b}} \mathbf{a}) \frac{1}{\|\mathbf{b}\|} \mathbf{b} = \frac{2}{3} \langle 1, 2, 2 \rangle \quad (15)$$

25. Repeat example 3.6 with an angle of  $\frac{\pi}{3}$  with the horizontal.

We have that  $\mathbf{F} = \langle 40 \cos(\frac{\pi}{3}), 40 \sin(\frac{\pi}{3}) \rangle = \langle 20, 20\sqrt{3} \rangle$ .

Also  $\mathbf{d} = \langle 5280, 0 \rangle$  So  $W = \mathbf{F} \cdot \mathbf{d} = 105600$ .

31. Label each statement as true or false, if true, explain why, if false give a counterexample.

a) If  $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$  then  $\mathbf{b} = \mathbf{c}$ . False let  $\mathbf{a} = \langle 4, 0, 2 \rangle$ ,  $\mathbf{b} = \langle 4, 0, 0 \rangle$  and  $\mathbf{c} = \langle 0, 0, 8 \rangle$ .

b) If  $\mathbf{b} = \mathbf{c}$  then  $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ . True.

c)  $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$ . True: If  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$  then  $\mathbf{a} \cdot \mathbf{a} = \sum_{i=1}^3 a_i^2 = \|\mathbf{a}\|^2$ .

d) If  $\|\mathbf{a}\| > \|\mathbf{b}\|$  then  $\mathbf{a} \cdot \mathbf{c} > \mathbf{b} \cdot \mathbf{c}$ . False take  $\mathbf{a} = \langle -8, -9, -10 \rangle$ ,  $\mathbf{b} = \langle 2, 2, 2 \rangle$  and  $\mathbf{c} = \langle 1, 1, 1 \rangle$ .

e) If  $\|\mathbf{a}\| = \|\mathbf{b}\|$  then  $\mathbf{a} = \mathbf{b}$ . False, take  $\mathbf{a} = \langle -2, 0, 0 \rangle$  and  $\mathbf{b} = \langle 2, 0, 0 \rangle$ .

33. By the Cauchy Schwartz Inequality  $|\mathbf{a} \cdot \mathbf{b}| \leq \|\mathbf{a}\| \|\mathbf{b}\|$ , What relationship must exist between  $\mathbf{a}$  and  $\mathbf{b}$  to have  $|\mathbf{a} \cdot \mathbf{b}| = \|\mathbf{a}\| \|\mathbf{b}\|$ .

They must be parallel for the  $\cos(\theta) = \pm 1$ .

37. For vectors  $\mathbf{a}$  and  $\mathbf{b}$  use the Cauchy Schwartz inequality to find the maximum value of  $\mathbf{a} \cdot \mathbf{b}$  when  $\|\mathbf{a}\| = 3$  and  $\|\mathbf{b}\| = 5$ .

The maximum value is  $\|\mathbf{a}\| \|\mathbf{b}\| = 15$ .

1. Write  $\mathbf{a} = \langle 8, 6, 8 \rangle$  in terms of the sum of a vector parallel to  $\mathbf{b} = \langle 1, 2, -1 \rangle$  and a vector perpendicular to  $\mathbf{b}$ .

$$\text{proj}_{\mathbf{b}} \mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \mathbf{b} = \frac{8 \cdot 1 + 6 \cdot 2 - 8}{1 + 4 + 1} \langle 1, 2, -1 \rangle \quad (16)$$

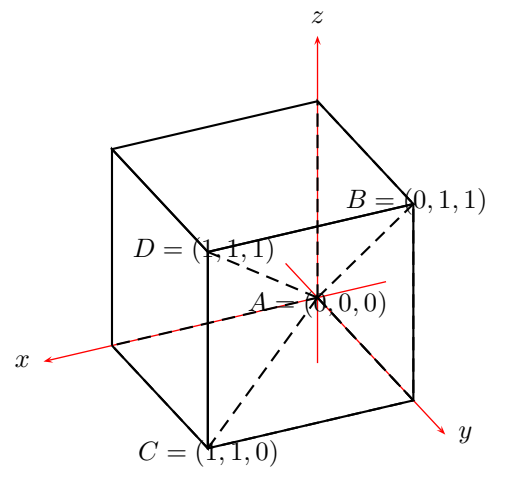
$$= 2 \langle 1, 2, -1 \rangle = \langle 2, 4, -2 \rangle \quad (17)$$

This vector is the vector parallel to  $\mathbf{b}$ . Now

$$\mathbf{a} - \langle 2, 4, -2 \rangle = \langle 8 - 2, 6 - 4, 8 + 2 \rangle = \langle 6, 2, 10 \rangle \quad (18)$$

Thus we have that

$$\mathbf{a} = \langle 8, 6, 8 \rangle = \langle 2, 4, -2 \rangle + \langle 6, 2, 10 \rangle \quad (19)$$



2. The picture is this

Take the face diagonals to be the vectors  $\overrightarrow{AB} = \langle 0, 1, 1 \rangle$  and  $\overrightarrow{AC} = \langle 1, 1, 0 \rangle$  then the main diagonals are represented by the vector  $\overrightarrow{AD} = \langle 1, 1, 1 \rangle$ . The angle between  $\overrightarrow{AB}$  and  $\overrightarrow{AC}$  is

$$\arccos\left(\frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\|\overrightarrow{AB}\| \|\overrightarrow{AC}\|}\right) = \arccos\left(\frac{1}{\sqrt{2} \cdot \sqrt{2}}\right) = \frac{\pi}{3} \quad (20)$$

The angle between  $\overrightarrow{AB}$  and  $\overrightarrow{AD}$  is

$$\arccos\left(\frac{\overrightarrow{AB} \cdot \overrightarrow{AD}}{\|\overrightarrow{AB}\| \|\overrightarrow{AD}\|}\right) = \arccos\left(\frac{2}{\sqrt{2} \cdot \sqrt{3}}\right) = \arccos\left(\frac{2}{\sqrt{6}}\right) \quad (21)$$