1. Let $w = f(x, y)$ where $x = u + v$ and $y = uv$.

a) Find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ in terms of partial derivatives of w with respect to x and y.

b) Find $\frac{\partial^2 w}{\partial u \partial v}$.

a) First use the chain rule for 1 variable namely

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}v \tag{1}$$

and also

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}u \tag{2}$$

b) For this part what we need to do is realize that $\frac{\partial^2 w}{\partial u \partial v} = \frac{\partial}{\partial u}\frac{\partial w}{\partial v}$ thus we must differentiate each object carefully. Start with $\frac{\partial w}{\partial x}$. We know that really this is $h(u, v) = \frac{\partial w}{\partial x}(x, y)$. Now we must take $\frac{\partial h}{\partial u}$ and use the chain rule for this. Using the chain rule this equals

$$\frac{\partial h}{\partial u} = \frac{\partial}{\partial u}\frac{\partial w}{\partial x} = \frac{\partial^2 w}{\partial x^2}\frac{\partial x}{\partial u} + \frac{\partial^2 w}{\partial y \partial x}\frac{\partial y}{\partial u}. \tag{3}$$

Then evaluating the first partials we get that this equals

$$\frac{\partial}{\partial u}\frac{\partial w}{\partial x} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y \partial x}v \tag{4}$$

Now if we take the right hand side of (2) if we want to take $\frac{\partial}{\partial u}\left(\frac{\partial w}{\partial y}u\right)$ then we must use both the product rule and the chain rule.

Using the product rule we get that

$$\frac{\partial}{\partial u}\left(\frac{\partial w}{\partial y}u\right) = \left(\frac{\partial}{\partial u}\frac{\partial w}{\partial y}\right)u + \frac{\partial w}{\partial y} \tag{5}$$

Now to get the last ingredient in our recipe for chain rule fun we must use the chain rule again, This time on the function $g(u,v) = \frac{\partial w}{\partial y}(x,y)$. This gives us the following

$$\frac{\partial g}{\partial u} = \frac{\partial}{\partial u}\frac{\partial w}{\partial y} = \frac{\partial^2 w}{\partial x \partial y}\frac{\partial x}{\partial u} + \frac{\partial^2 w}{\partial y^2}\frac{\partial y}{\partial u} \tag{6}$$

then adding in the first partials we get that this equals

$$\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial y^2} v \tag{7}$$

Taking all of the ingredients together and mixing gives us that

$$\frac{\partial^2 w}{\partial u \partial v} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y \partial x} v + \frac{\partial^2 w}{\partial x \partial y} u + \frac{\partial^2 w}{\partial y^2} uv + \frac{\partial w}{\partial y} \tag{8}$$

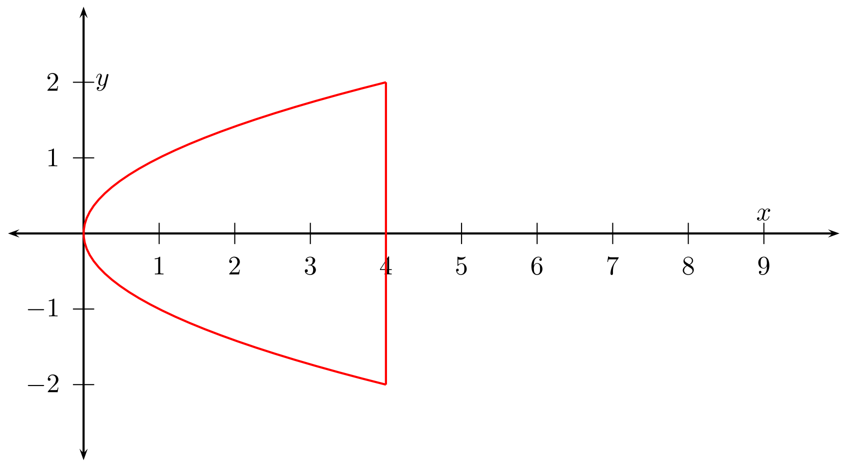
If conditions on f are nice enough we have that $\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial^2 w}{\partial y \partial x}$ and thus that

$$\frac{\partial^2 w}{\partial u \partial v} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial x \partial y}(u+v) + \frac{\partial^2 w}{\partial y^2} uv + \frac{\partial w}{\partial y} \tag{9}$$

Now let us do some double integrals

Problem 26 asks us to calculate the mass and the center of mass for the lamina bounded by $x = y^2$ and $x = 4$ with the given density $\rho(x,y) = y + 3$.

We are in the following region enclosed in red



Let us take M the center of mass given on this region by

$$M = \iint_R \rho(x,y)dA = \int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} (y+3)dxdy = \int_0^4 \frac{y^2}{2} + 3y|_{-\sqrt{x}}^{\sqrt{x}} \tag{10}$$

Taking (10) we get that this equals

$$M = \int_0^4 \Big((\frac{x}{2} + 3\sqrt{x}) - (\frac{x}{2} - 3\sqrt{x}) \Big) dx = \int_0^4 6\sqrt{x}dx = 4x^{\frac{3}{2}}|_0^4 = 32. \tag{11}$$

Then we have that for M_x we have that

$$M_x = \iint_R y\rho(x,y)dA = \int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} (y^2+3y)dydx = \int_0^4 \Big(\frac{y^3}{3} + \frac{3}{2}y^2|_{-\sqrt{x}}^{\sqrt{x}} \Big) dx \tag{12}$$

reducing (12) we find that

$$M_x = \int_0^4 \frac{2}{3}x^{\frac{3}{2}}dx = \frac{4}{15}x^{\frac{5}{2}}|_0^4 = \frac{128}{15} \tag{13}$$

For M_y we then similarly compute that

$$M_y = \iint_R x\rho(x,y)dA = \int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} (xy+3x)dydx = \int_0^4 x\frac{y^2}{2} + 3xy|_{-\sqrt{x}}^{\sqrt{x}} dx \quad (14)$$

expanding (14) we find that it equals

$$M_y = \int_0^4 \Big((x^2 + 3x^{\frac{3}{2}}) - (x^2 - 3x^{\frac{3}{2}})\Big)dx = \int_0^4 6x^{\frac{3}{2}}dx = \frac{12}{5}x^{\frac{5}{2}}|_0^4 = \frac{384}{5} \quad (15)$$

Putting (11), (13) and (15) we have that the center of mass is $(\frac{M_y}{M}, \frac{M_x}{M}) = (\frac{12}{5}, \frac{4}{5})$.

Hopefully we all understand lamina and double integrals better now.