Homotopy invariance of winding numbers—the continuous case

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The winding number as $\sum \Delta\theta$.

Although there is no continous function

$$\theta\colon \mathbb{R}^2\backslash\{P\} \to \mathbb{R}$$

whose value gives the angle from due east, there is such a function on any sector whose angle is less than 2π.

Lemma 1. *If $\gamma\colon [a,b] \to \mathbb{R}^2\backslash\{P\}$ is a continous path, then it one can divide the interval into subdivisions*

$$a = t_0 \le t_1 \le \cdots \le t_n = b$$

such that each $\gamma([t_i, t_{i+1}])$ is contained in a sector. □

Choose such a subdivision, and for $1 \le i \le n$, let θ_i be an angle function defined on the sector containing $\gamma([t_{i-1}, t_i])$. Define

$$W(\gamma, P) = \sum_{i=1}^{n} \theta_i(\gamma(t_i)) - \theta_i(\gamma(t_{i-1})).$$

One shows (3.1, 3.2) that this is independent of the choices made to define it, and that it extends the smooth notion of winding number.

Subdivision.

How can we prove without Green's theorem that $W(\gamma, P)$ is a homotopy invariant? The essential technique is "subdivision".

It suffices to consider the following situation. Suppose that $R = [a, b] \times [c, d]$ is a rectangle, and

$$\Gamma \colon R \to \mathbb{R}^2 \backslash \{P\}$$

is a (continuous) map. Let

$$\begin{aligned} \gamma_1(s) &= \Gamma(s, c) \colon [a, b] \to \mathbb{R}^2 \backslash \{P\} \\ \gamma_2(t) &= \Gamma(b, t) \colon [c, d] \to \mathbb{R}^2 \backslash \{P\} \\ \gamma_3(s) &= \Gamma(a + b - s, d) \colon [a, b] \to \mathbb{R}^2 \backslash \{P\} \\ \gamma_4(t) &= \Gamma(a, c + d - t) \colon [c, d] \to \mathbb{R}^2 \backslash \{P\}. \end{aligned}$$

Lemma 2. *If* $\Gamma(R)$ *is contained in a sector, then*

$$\sum_i W(\gamma_i, P) = 0. \quad \square$$

Proposition 1. *For* any R,

$$\sum_i W(\gamma_i, P) = 0.$$

Dependence on P

Let $\gamma\colon [a,b] \to \mathbb{R}^2$ be a closed path. For $P \notin \gamma([a,b])$, we can define $W(P,\gamma) \in \mathbb{Z}$.

Lemma 3. *The function*

$$W(-,\gamma)\colon \mathbb{R}^2 \backslash \gamma([a,b]) \to \mathbb{Z}$$

is locally constant.

Degree

If f and g are two maps $X \to Y$, then a homotopy from f to g is a map $H\colon [0,1] \times X \to Y$ such that

$$\begin{aligned} H(0,x) &= f(x) \\ H(1,x) &= g(x). \end{aligned}$$

If (X,p) and (Y,q) are pointed spaces, and f and g are pointed maps from X to Y, then a pointed homotopy from f to g is a homotopy H from f to g which also satisfies

$$H(s,p) = q$$

for all $s \in [0,1]$.

It is easy to check that homotopy is an equivalence relation. Let

$$[(X,p),(Y,q)]$$

denote the pointed homotopy classes of maps from (X,p) to (Y,q).

In 3d you show that

Proposition 2. *The map* $f \mapsto \deg f$ *is an isomorphism*

$$[(C, c), (C', c')] \cong \mathbb{Z}$$

for any pointed circles (C, c) *and* (C', c') *in* $\mathbb{R}^2$.

Let $f\colon C \to C'$ be a map. To define its degree, pick a point P inside C'. Let $\pi\colon [a, b] \to C$ be a map from an interval to C, such that

$$\pi(a) = \pi(b)$$

and $\pi|_{(a,b)}$ is a homeomorphism onto its image. Define

$$\deg f = W(f\pi, P).$$

A very typical result is the following. Let D be the disk whose boundary is C.

Lemma 4. *If* f *extends to a map* $F\colon D \to C'$, *then* $\deg f = 0$.