

Elliptic curves and algebraic topology

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Elliptic curves enter algebraic topology through “Elliptic cohomology”—really a family of cohomology theories—and their associated “elliptic genera”.

- Arithmetic aspect: Modularity of elliptic genera, The spectrum TMF of “topological modular forms” and the calculation of $\pi_*TMF \rightarrow MF(\mathbb{Z})$, Hopkins’s proof of Borcherds’ congruences.
- Physical aspect: Witten’s approach to elliptic genera via string theory.
- Homotopy theoretic aspect: Relationship to chromatic program, Hopkins and Mahowald’s calculation of $\pi_*S \rightarrow \pi_*TMF$.

Forgetful functor
Not quite faithful. Instead
Adams Spectral Sequence

The Adams spectral sequence and descent

Let E be a generalized (co)homology theory. One could hope to study maps from X to Y by studying maps

$$E_*X \rightarrow E_*Y$$

(of E_*E -comodules).

For example, consider

$$S^1 \xrightarrow{n} S^1$$

This is detected perfectly well by $E = H\mathbb{Z}$; when $E = H\mathbb{Z}/2$, at least it captures the parity of n .

For $E = H\mathbb{Z}/2$, $n = 2$, one uses the cofiber sequence

$$S^1 \xrightarrow{2} S^1 \rightarrow M = e^2 \cup_2 S^1 \rightarrow S^2;$$

this gives a short exact sequence

$$E_*S^1 \rightarrow E_*M \rightarrow E_*S^2$$

i.e. an element of

$$\mathrm{Ext}_{E_*E}^1(E_*S^1, E_*S^2) = \mathrm{Ext}_{E_*E}^{1,1}(E_*S^0, E_*S^0)$$

Note that considering E_*E comodules distinguishes M from $S^1 \vee S^2$, which corresponds to the zero map $S^1 \rightarrow S^1$.

The *Adams spectral sequence* is of the form

$$E_2 = \text{Ext}_{E_*E}^{s,t}(E_*S^0, E_*X) \Rightarrow \pi_{t-s}L_EX.$$

The exact couple can be obtained from the sequence

$$S \longrightarrow E \rightrightarrows E \wedge E \rightrightarrows E \wedge E \wedge E \dots$$

This is analogous to the sequence

$$S \longrightarrow E \rightrightarrows E \otimes_S E \rightrightarrows E \otimes_S E \otimes_S E$$

for faithfully flat descent: if $S \rightarrow E$ is a faithfully flat map of commutative rings, then the forgetful functor

$(S\text{-modules}) \rightarrow (E\text{-modules with descent data})$

is an equivalence of categories.

After applying homotopy groups, one has

$$\pi_* S \longrightarrow \pi_* E \rightrightarrows E_* E \rightrightarrows E_* E \otimes_{E_*} E_* E$$

This is an example of a “Hopf algebroid”. Let $A = E_*$ and $\Gamma = E_* E$; then the basic object of study is

$$A \rightrightarrows \Gamma \rightrightarrows \Gamma \otimes_A \Gamma$$

Geometric reformulation

A Mittag-Leffler sequence is a sequence of epimorphisms.—A. Neeman.

In the language of algebraic geometry, a Hopf algebroid is a group acting on a space

$$G \times G \times T \rightrightarrows G \times T \xrightarrow[\pi]{\mu} T \dashrightarrow T/G$$

and a comodule is an equivariant sheaf. That is

$$\begin{aligned} \mathrm{sp} E_* &= T \\ \mathrm{sp} E_* E &= G \times T \\ \widetilde{E_* X} &= V; \end{aligned}$$

and the E_2 -term of the Adams spectral sequence is just

$$\mathrm{Ext}_{E_* E}^*(E_*, E_* X) = H^*(G; V),$$

the G -equivariant cohomology of the sheaf V , which is the natural object to study for questions of descent to T/G .

The Adams-Novikov spectral sequence

Suppose that E is *complex oriented*: that is, it comes with an isomorphism

$$E^*\mathbb{C}P^\infty \cong E^*[[x]],$$

and so the map

$$\mathbb{C}P^\infty \times \mathbb{C}P^\infty \rightarrow \mathbb{C}P^\infty$$

classifying the tensor product of complex line bundles gives rise to a map

$$E^*[[x]] \cong E^*\mathbb{C}P^\infty \rightarrow E^*\mathbb{C}P^\infty \times \mathbb{C}P^\infty \cong E^*[[x, y]],$$

that is, a *formal group law* over E .

The initial example of a complex oriented cohomology theory is MU , complex cobordism. Quillen showed that π_*MU carries the *universal* formal group law; and it turns out that MU_*MU represents the group of isomorphisms of formal group laws. That is,

$$\begin{aligned} \mathrm{sp} \pi_*MU &= FGL \\ \mathrm{sp} MU_*MU &= FGLI \times FGL. \end{aligned}$$

Thus the E_2 -term of the MU -Adams spectral sequence (“Adams-Novikov spectral sequence”) for X is

$$\text{Ext}_{MU_*MU}(\pi_*MU; MU_*X) \cong H^*(FGLI; \widetilde{MU_*X});$$

it's calculating descent for

$$FGLI \times FGL \begin{array}{c} \xrightarrow{\mu} \\ \xrightarrow{\pi} \end{array} FGL \dashrightarrow (\text{formal groups})$$

Morava took this picture seriously, and Devinatz, Hopkins, Landweber, Miller, Ravenel, Smith, Wilson... took him seriously, and so we have chromatic stable homotopy theory. Note that this is saying that the differentials in the Adams-Novikov spectral sequence are measuring the difference between the (homotopy groups of) spheres and (the functions on) the space of formal groups.

Locally at a prime p , a formal group has an important invariant called the *height*: it's a non-negative integer. Associated to each height is a family of cohomology theories which detect the associated chromatic homotopy theory.

height	0	1	2	n
Morava	$H\mathbb{Q}$	K/p	Ell/p	$K(n)$
Lubin-Tate	$H\mathbb{Q}$	K_p	Ell_p	E_n
Global	$H\mathbb{Q}$	K	Ell	?
Formal group	\mathbb{G}_a	\mathbb{G}_m	\hat{C}	Honda

For example, localizing to a neighborhood of a height n formal group Γ replaces MU with E_n and $FGLI$ with $\mathbb{S}_n = Aut(\Gamma)$, and the Adams spectral sequence becomes

$$H^*(\mathbb{S}_n; E_{n*}X) \Rightarrow \pi_* L_{K(n)}X.$$

For $n = 1$ this spectral sequence is well understood and describes the part of $\pi_* S$ detected by the J -homomorphism. For $n \geq 2$ the situation is more complicated.

Elliptic cohomology

An *elliptic cohomology theory* is a commutative, even periodic cohomology theory, whose formal group is given with an isomorphism to the formal group of an elliptic curve. The formal groups of (generalized) elliptic curves have height (0 and) 1 and 2.

Two big advantages of elliptic cohomology (over E_2 , say).

1) Elliptic curves are projective algebraic objects; for example this helps with calculation: for example, the sigma orientation and the calculation by Hopkins and Mahowald of $\pi_*TMF \rightarrow MF(\mathbb{Z})$.

2) Elliptic cohomology appears to offer a relationship to geometry and analysis.

What is TMF ?

The definition of elliptic cohomology suggests the question: which elliptic curves admit elliptic cohomology theories? Can the construction be made functorial?

Hopkins and Miller have constructed a functor

$$C : (\text{étale elliptic curves}) \rightarrow (E_\infty \text{ ring spectra})$$

The spectrum of *topological modular forms* is the homotopy inverse limit

$$TMF = \text{holim } C.$$

It turns out that TMF is not an elliptic spectrum, but by construction there is a canonical map from TMF to any E_∞ elliptic spectrum.

In order to give moduli for the space of formal groups, it was necessary to describe the space of formal groups as the quotient

$$(\text{formal groups}) \cong FGL/FGLI.$$

This is also the case for the space of elliptic curves: every elliptic curve is isomorphic to a Weierstrass curve, i.e. one of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

and two Weierstrass cubics give the same elliptic curve if they differ by

$$x' = u^2x + rx' = u^3y + sx + t.$$

The space of elliptic curves is the quotient

$$ELL \cong W/WI.$$

The dual Lie algebra of the universal elliptic curve over W defines an equivariant sheaf ω over W . By construction there is a spectral sequence with

$$E_2 = H^s(WI; \omega^t) \Rightarrow \pi_{t-s}TMF.$$

The name “topological modular forms” comes from the fact that

$$H^0(WI; \omega^t)$$

is the group of integral modular forms of weight $\pm t$. Indeed the edge homomorphism gives a map

$$\pi_* TMF \rightarrow MF_*(\mathbb{Z}).$$

Hopkins and Mahowald have calculated this map completely.

E_∞ haiku

Cochain cup product
Not quite commutative
Steenrod operations

Geometric and Physical Aspect

In any cohomology theory E one might try to study the *Umkehr* map. If $f : X \rightarrow Y$ is a proper map with fiber dimension d , then this is a homomorphism

$$f_! : E^* X \rightarrow E^{*-d} Y.$$

The construction depends on the choice of an orientation in E -theory for the bundle T_f of tangents along the fibers of f . An important special case is the map

$$\pi^X : X \rightarrow *;$$

by applying $\pi_!$ to various characteristic classes of X you get various “genera” associated to E -theory.

The choice of orientation is really the crux of the matter: for example, a consistent family of orientations in E -theory for spin vector bundles is equivalent to a map of ring spectra

$$MSpin \rightarrow E,$$

and the ring homomorphism

$$\pi_* MSpin \rightarrow \pi_* E$$

sends X to $\pi_!^X(1)$.

The interpretation of this map often offers insight into the geometry and analysis associated to E . For example

H integration along the fiber
 K index of family of elliptic operators
 Ell one-loop amplitude of string theories.

The calculation of elliptic genera using string theory is more than an idle observation: it leads to physical proofs of mathematical results: particularly, the modularity and rigidity of the Witten genus

$$w : \pi_* MSpin \rightarrow \mathbb{Z}[[q]].$$

Let M be a spin manifold with Dirac operator D . Let T be its tangent bundle. Let V be another spin vector bundle, and let $\Delta_{-1}V$ be the associated spinor bundle. Let

$$S_t : KO(X) \rightarrow K(X)[[t]]$$

be the exponential characteristic class

$$S_t W = \sum_{k \geq 0} t^k S^k(W \otimes \mathbb{C})$$

and similarly for Λ . If W is a vector bundle, let \overline{W} be the associated reduced bundle. Then

$$\begin{aligned} w(M; V) = & \text{ind}(D \otimes \bigotimes_{n \geq 0} S_{q^n} \overline{T} \otimes \Delta_{-1} V \otimes \bigotimes_{n \geq 0} \Lambda_{q^n} \overline{V}) \\ & \in \mathbb{Z}[[q]]. \end{aligned}$$

Witten gave physical proofs of two results about w .

Proposition (Modularity). *If $c_2(T - V) = 0$, then $w(M; V)$ is the q -expansion of a modular form.*

If S^1 acts on the whole situation, then we can consider the equivariant Witten genus

$$w_{S^1}(M; V) \in (\mathbb{Z}[\lambda, \lambda^{-1}])[[q]].$$

Proposition (Rigidity). *If $w_2(T - V)_{S^1} = 0$ and $c_2(T - V)_{S^1} = 0$, then for all k the coefficient of q^k in $w_{S^1}(M; V)$ is a constant Laurent polynomial; that is,*

$$w_{S^1}(M; V) = w(M; V).$$

To explain the modularity of the Witten genus, we locate it in elliptic cohomology. To begin with, there is an elliptic cohomology theory based on the Tate elliptic curve; it was discovered by Morava in the early 70's. Because the formal group of the Tate curve is the multiplicative formal group, it is a form of K -theory, with coefficients extended to power series in q .

Hopkins, Rezk, and I have constructed a map

$$\sigma : MO\langle 8 \rangle \rightarrow TMF,$$

such that the diagram

$$\begin{array}{ccc} MO\langle 8 \rangle & \longrightarrow & TMF \\ \downarrow & & \downarrow \\ MSpin & \xrightarrow{w} & K[[q]] \end{array}$$

commutes.

It follows that the Witten genus of an $MO\langle 8 \rangle$ manifold is the q -expansion of a modular form. Indeed it is in the image of

$$\pi_* tmf \rightarrow MF(\mathbb{Z}) \rightarrow \mathbb{Z}[[q]].$$

Hopkins and Mahowald have shown that this map has a cokernel, which corresponds to divisibility results for Witten genera.

Similarly, the kernel (which is all 24-torsion) gives new torsion invariants for $MO\langle 8 \rangle$ manifolds.

Plan.

In the next lecture, I shall discuss the sigma orientation

$$MO\langle 8 \rangle \rightarrow Tmf. \quad (1)$$

In fact I shall discuss the more elementary construction of a map

$$MU\langle 6 \rangle \rightarrow E$$

from $MU\langle 6 \rangle$ to any elliptic cohomology theory. The construction illustrates the power of working with elliptic curves instead of mere formal groups.

In the third lecture, I shall explain how the Rigidity Theorem follows from the analogue of (1) in (analytic) S^1 -equivariant elliptic cohomology.