Modularity in elliptic cohomology

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In the previous lecture I introduced an invariant of spin manifolds called the *Witten genus*.

Let M be a spin manifold with Dirac operator D. Let T be its tangent bundle. Let V be another spin vector bundle, and let $\Delta_{-1}V$ be the associated spinor bundle. Let

$$S_t : KO(X) \to K(X)[\![t]\!]$$

be the exponential characteristic class

$$S_t W = \sum_{k \geq 0} t^k S^k(W \otimes \mathbb{C})$$

and similarly for Λ. If W is a vector bundle, let $\overline{W}$ be the associated reduced bundle. Then

$$w(M;V) = \\ \mathrm{ind}(D \otimes \bigotimes_{n \geq 1} S_{q^n}\overline{T} \otimes \Delta_{-1}V \otimes \bigotimes_{n \geq 1} \Lambda_{q^n}\overline{V}) \\ \in \mathbb{Z}[\![q]\!].$$

Taking $V = 0$ gives the Witten genus; it is a ring homomorphism

$$w : \pi_* MSpin \to \mathbb{Z}[\![q]\!]$$

Witten gave physical proofs of two results about w.

Proposition (Modularity). *If* $c_2(T - V) = 0$*, then* $w(M; V)$ *is the* q*-expansion of a modular form.*

If S^1 acts on the whole situation, then we can consider the equivariant Witten genus

$$w_{S^1}(M; V) \in (\mathbb{Z}[\lambda, \lambda^{-1}])[\![q]\!].$$

Proposition (Rigidity). *If* $w_2(T - V)_{S^1} = 0$ *and* $c_2(T - V)_{S^1} = 0$*, then for all* k *the coefficient of* q^k *in* $w_{S^1}(M; V)$ *is a constant Laurent polynomial; that is,*

$$w_{S^1}(M; V) = w(M; V).$$

I explained that the modularity of the Witten genus is an easy consequence of the existence of a map of ring spectra $\sigma : MO\langle 8\rangle \to TMF$ such that the diagram

$$\begin{tikzcd}
MO\langle 8\rangle \arrow[r, "\sigma"] \arrow[d] & TMF \arrow[d] \\
MSpin \arrow[r, "w"] & > KO
\end{tikzcd}$$

commutes.

In this lecture I shall explain how to construct a natural map

$$\sigma(E, C, t) : MU\langle 6\rangle \to E$$

from the bordism spectrum of stably complex manifolds with trivialization of c_1 and c_2 to any elliptic spectrum, such that the diagram

$$\begin{tikzcd}
MU\langle 6\rangle \arrow[r, "\sigma(K_{\mathrm{Tate}})"] \arrow[d] & K[\![q]\!] \\
MSU \arrow[r] & MSpin \arrow[u, "w"']
\end{tikzcd}$$

commutes. Already this proves that the Witten genus of any $BU\langle 6\rangle$ manifold is the q-expansion of a modular form.

The proof will eventually reduce to the observation that if

$$\sigma(L,q) = (L^{1/2} - L^{-1/2}) \prod_{n\geq 1} \frac{(1-q^nL)(1-q^nL^{-1})}{(1-q^n)^2},$$

then on the one hand $\sigma(L,q)$ is just the product formula for the Weierstrass sigma function, and on the other hand if we write $T = TM$ as a sum of complex line bundles

$$T = L_1 + \cdots + L_r,$$

then

$$\Delta_{-1}(T) \otimes \bigotimes_{n\geq 1} S_{q^n}(\overline{T}) \cong \prod_{j=1}^{r} \sigma(L_i,q)^{-1}.$$

Genera after Hirzebruch

Let R be a $\mathbb{Q}$-algebra. To give a ring homomorphism

$$MU_* \xrightarrow{\phi} R,$$

it is equivalent to give a power series

$$f(x) = x + o(x^2) \in R[\![x]\!].$$

If ϕ is such a genus, then the associated power series is determined by

$$f(x)^{-1} = \sum_{n \geq 0} \frac{\phi(\mathbb{C}P^n)}{n+1} x^{n+1}. \tag{1}$$

Let f be a power series. If M is a stably complex manifold of dimension $2r$, and if we use the splitting principle to write

$$TM = L_1 + \cdots + L_r,$$

and set $x_i = c_1 L_i$, then

$$\phi(M) = \int_M \prod_{j=1}^{r} \frac{x_i}{f(x_i)}.$$

For example, the genus assoicated to the power series

$$f(x) = 1 - e^{-x}$$

is called the Todd genus; equation (1) reflects the fact that Todd($\mathbb{C}P^n$) = 1 for all n.

Genera after Dold and Adams

Let E be a commutative ring spectrum. Then maps of ring spectra

$$MU \to E$$

are in bijective correspondence with classes

$$u \in E^2(\mathbb{C}P^\infty)$$

such that

$$u|_{\mathbb{C}P^1} = \Sigma^2(1)$$

in $E^2(\mathbb{C}P^1)$.

Given such a u, the associated genus

$$\phi : \pi_* MU \to \pi_* E$$

is given by the formula

$$\phi(M) = \pi_!^M(1),$$

where $\pi_!^M$ is the Umkehr map

$$E^*(M) \to E^{*-d}(*)$$

associated to the map $\pi^M : M \to *$ and the orientation u.

Consider

$$E^*(\mathbb{C}P^\infty) \xrightarrow{c} (E \wedge H\mathbb{Q})^*(\mathbb{C}P^\infty) \leftarrow H\mathbb{Q}^*(\mathbb{C}P^\infty).$$

If $x = c_1 L$, then we can write

$$c(u) = f(x) \in (E \wedge H\mathbb{Q})^*[\![x]\!].$$

The Riemann-Roch formula is

$$\pi_!^M(1) = \int_M \prod_j \frac{x_j}{f(x_j)},$$

where x_j are the roots of the total Chern class of M.

For example, if $E = K$ and $v \in K^0(\mathbb{C}P^1) \cong K^{-2}(*)$ is the Bott element, then

$$c : K \to K \wedge H\mathbb{Q} \cong H\mathbb{Q}[v, v^{-1}]$$

is the Chern character, and

$$u = v^{-1}(1 - L) \in K^2(\mathbb{C}P^\infty)$$

gives an orientation

$$MU \to K,$$

such that

$$c(u) = v^{-1}(1 - e^{vx}),$$

and so

$$\pi_!^M(1) = v^r \,\mathrm{Todd}(M).$$

where $\dim M = 2r$.

Dold-Adams II

Let L be the tautological line bundle over $\mathbb{C}P^\infty$. The map $\mathbb{C}P^\infty \to BU$ classifying the reduced bundle $L-1$ gives a map of Thom spectra

$$\Sigma^{-2}(\mathbb{C}P^\infty)^L \cong (\mathbb{C}P^\infty)^{L-1} \xrightarrow{j} MU.$$

Restricting to the basepoint of $\mathbb{C}P^\infty$ gives

$$S^0 \to (\mathbb{C}P^\infty)^{L-1} \xrightarrow{j} MU.$$

Proposition. *If E is a commutative ring spectrum, then restriction along j gives an isomorphism*

$$\text{RingSpectra}(MU, E) \cong \pi_0 \left(\begin{tikzcd} (\mathbb{C}P^\infty)^{L-1} \arrow[r, dashed] & E \\ S^0 \arrow[u] \arrow[ur, "\eta"'] & \end{tikzcd} \right)$$

The homeomorphism $(\mathbb{C}P^\infty)^L \cong \mathbb{C}P^\infty$ identifies the right-hand side with the set of $u \in E^2(\mathbb{C}P^\infty)$ such that $u|_{\mathbb{C}P^1} \cong \Sigma^2(1)$.

Algebro-geometric formulation

Let E be a commutative, even periodic ring spectrum. Then $E^0\mathbb{C}P^\infty$ is the ring of functions on a formal group G over $S_E = \operatorname{sp}\pi_0 E$. The augmentation

$$E^0\mathbb{C}P^\infty \to \pi_0 E$$

corresponds to the identity section $S_E \xrightarrow{0} G$.

The zero section

$$\mathbb{C}P^\infty \to (\mathbb{C}P^\infty)^L$$

identifies $E^0((\mathbb{C}P^\infty)^L)$ with the ideal $I_G(0)$ of functions on G which vanish at 0. The restriction

$$E^0((\mathbb{C}P^\infty)^L) \to E^0 S^2$$

identifies $E^0 S^2 \cong \pi_0 E$ with

$$\omega \overset{\text{def}}{=} 0^* I_G(0) \cong T_0^* G,$$

the cotangent space of G at the origin, and so $E^0 S^{-2} \cong \omega^{-1}$.

Thus

$$E^0((\mathbb{C}P^\infty)^{L-1})$$

is the $E^0\mathbb{C}P^\infty$-module of sections of $I_G(0) \otimes p^*\omega^{-1}$. Notice that

$$0^*(I_G(0) \otimes p^*\omega^{-1}) \cong \omega \otimes \omega^{-1}$$

has a canonical trivialization; this corresponds to the fact that

$$E^0(\Sigma^{-2}S^2) \cong E^0(S^0) \ni 1.$$

Proposition. *The set of maps of ring spectra* $MU \to E$ *is in bijective correspondence with the set of rigid sections of* $I_G(0) \otimes p^*\omega^{-1}$*.*

MSU and $MU\langle 6\rangle$

Let V_k be the virtual bundle

$$V_k = \prod_{j=1}^{k} (1 - L_j)$$

over $(\mathbb{C}P^\infty)^k$. It turns out that the map

$$(\mathbb{C}P^\infty)^k \to BU$$

classifying this bundle factors uniquely through $BU\langle 2k\rangle$, and so we get a map of Thom spectra

$$(\mathbb{C}P^{\infty k})^{V_k} \to MU\langle 2k\rangle.$$

It turns out that $E^0((\mathbb{C}P^{\infty k})^{V_k}))$ is the module of sections of $\Theta^k(I_G(0))$. Θ^k is most easily illustrated by example: if $\mathcal{L}$ is a line bundle over G, then $\Theta^k(\mathcal{L})$ is a line bundle over G^k

given by

$$\begin{aligned}\Theta^0(\mathcal{L}) &= \mathcal{L} \\ \Theta^1(\mathcal{L})_a &= \frac{\mathcal{L}_a}{\mathcal{L}_0} \\ \Theta^2(\mathcal{L})_{a,b} &= \frac{\mathcal{L}_{a+b}\mathcal{L}_0}{\mathcal{L}_a\mathcal{L}_b} \\ \Theta^3(\mathcal{L})_{a,b,c} &= \frac{\mathcal{L}_{a+b+c}\mathcal{L}_a\mathcal{L}_b\mathcal{L}_c}{\mathcal{L}_{a+b}\mathcal{L}_{a+c}\mathcal{L}_{b+c}\mathcal{L}_0}.\end{aligned}$$

Notice that for $k \geq 1$, $\Theta^k(\mathcal{L})$ is

1. *rigid*: there is a canonical isomorphism
$$\Theta^k(\mathcal{L})_0 \cong \mathbf{1}$$
2. *symmetric*: for $\sigma \in \Sigma_k$, there is a canonical isomorphism
$$\sigma^*\Theta^k(\mathcal{L}) \cong \Theta^k(\mathcal{L})$$

In addition, for $k \geq 2$, Θ^k satisfies a cocycle condition. A Θ^k-structure on $\mathcal{L}$ is a trivialization of $\Theta^k(\mathcal{L})$ which is a rigid symmetric cocycle.

Theorem. *Restriction to* $(\mathbb{C}P^{\infty k})^{V_k}$ *gives a map*

$$\text{RingSpectra}(MU\langle 2k\rangle, E) \to \\ (\Theta^k\text{-}\textit{structures on } I_G(0)).$$

For $1 \le k \le 3$, *this is an isomorphism.*

Suppose that we've chosen an isomorphism

$$E^0\mathbb{C}P^\infty \cong E^0[\![x]\!].$$

Then for $k = 3$, the theorem identifies the set of maps of ring spectra

$$MU\langle 6\rangle \to E$$

with the set of power series

$$f(x, y, z) \in E^0[\![x, y, z]\!]$$

which are symmetric in x, y, x and satisfy

$$f(x, y, z) = xyz + \text{higher terms}$$
$$f(x, y, z)f(w, x + y, z) = f(w + x, y, z)f(w, x, z),$$

where the sum inside parentheses uses the group law of $E^0\mathbb{C}P^\infty$. The set of such f are called *cubical structures* on $I_G(0)$.

To calculate the corresponding genus, find a power series $g(x) = x + o(x^2)$ such that

$$f(x, y, z) = \frac{g(x+y+z)g(x)g(y)g(z)}{g(x+y)g(x+z)g(y+z)}.$$

The genus is

$$\int_M \prod_j \frac{x_j}{g(x_j)}.$$

The sigma orientation

Theorem (Abel). *Let C be an elliptic curve, that is, a pointed proper smooth curve whose geometric fibers have genus* 1*. Then C has a unique structure of group such that $I_C(0)$ has a cubical structure. Every map $C \to C'$ of elliptic curves is a homomorphism of groups.*

Now let (E, C, t) be an elliptic spectrum. Abel's Theorem gives a cubical structure $s(C)$ on $I_C(0)$, and $t^* s(C)|_{\widehat{C}}$ gives a cubical structure on $I_G(0)$, and so a map of ring spectra

$$MU\langle 6\rangle \to E.$$

Abel's Theorem is characterizes the group structure of an ellliptic curve by saying that given points y and z in C, there is a meromorphic function on C with divisor $(0) + (-y - z) - (-y) - (-z)$. If $C = \mathbb{C}/\Lambda$, then this function is a scalar multiple of

$$\frac{\sigma(x+y+z)\sigma(x)}{\sigma(x+y)\sigma(x+z)}.$$

The cube structure arises from symmetrizing this to get

$$\frac{\sigma(x+y+z)\sigma(x)\sigma(y)\sigma(z)}{\sigma(x+y)\sigma(x+z)\sigma(y+z)}.$$

This shows that, in the case of the Tate curve, the sigma orientation is the restriction of the Witten genus.