Equivariant elliptic cohomology and rigidity

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Let M be a spin manifold with tangent bundle T and Dirac operator D. Let V be a spin vector bundle of even rank over M. Suppose moreover that the circle S^1 acts on the whole situation. Then the equivariant Witten genus of M twisted by V is

$$\begin{aligned} w(M;V)_{S^1} = \mathrm{ind}_{S^1}(D\otimes & \bigotimes_{n\geq 1} S_{q^n}(\overline{T})\otimes \\ & \Delta_{-1}(V)\otimes \\ & \bigotimes_{n\geq 1} \Lambda_{q^n}(\overline{V})) \in (R(S^1))[\![q]\!]. \end{aligned}$$

We fix an isomorphism $R(S^1) \cong \mathbb{Z}[\lambda, \lambda^{-1}]$. Thus

$$w(M;V)_{S^1} = \sum_k a_k(\lambda) q^k$$

is a power series in q, with each coefficient itself a Laurent polynomial. Witten gave a physical proof of the following

Theorem (Ridigity). *If the equivariant characteristic classes* $w_2(V-T)_{S^1}$ *and* $c_2(V-T)_{S^1}$ *vanish, then each* $a_k(\lambda)$ *is constant, that is,*

$$w(M;V)_{S^1} = w(M;V).$$

The rigidity theorem was first proved by Bott and Taubes, and Kefeng Liu gave another proof. I shall explain how this result follows from the existence of a Thom isomorphism in Grojnowski's equivariant elliptic cohomology, extending the sigma orientation. The idea that there should be such a proof is due to Haynes Miller, and the first proof along these lines was given by Ioanid Rosu, although for the Ochanine genus, whose rigidity does not require any condition on c_2.

Equivariant elliptic cohomology

It is familiar that Atiyah-Segal completion can be written

$$\begin{array}{lcccccc}
\operatorname{sp} K_{S^1}(*) & \cong & \mathbb{G}_m & \leftarrow & \widehat{\mathbb{G}}_m & \cong & \operatorname{spf} K(BS^1) \\
\operatorname{sp} K_T(*) & \cong & \check{T} \otimes \mathbb{G}_m & \leftarrow & \check{T} \otimes \widehat{\mathbb{G}}_m & \cong & \operatorname{spf} K(BT)
\end{array}$$

$$\operatorname{sp} K_G(*) \cong (\check{T} \otimes \mathbb{G}_m)/W \leftarrow (\check{T} \otimes \widehat{\mathbb{G}}_m)/W \cong \operatorname{spf} K(BG)$$

In the second line, T is a torus and $\check{T} = \hom(S^1, T)$ is its lattice of cocharacteris; in the third line, G is a connected compact Lie group with maximal torus T and Weyl group W. One might hope that if (E, C, t) is an elliptic spectrum, then there is an equivariant elliptic cohomology with a completion map whose values are given by the table

$$\begin{array}{lcccccc}
E_{S^1}(*) & \cong & C & \leftarrow & \widehat{C} & \cong & \operatorname{spf} E(BS^1) \\
E_T(*) & \cong & \check{T} \otimes C & \leftarrow & \check{T} \otimes \widehat{C} & \cong & \operatorname{spf} E(BT)
\end{array}$$

$$E_G(*) \cong (\check{T} \otimes C)/W \leftarrow (\check{T} \otimes \widehat{C})/W \cong \operatorname{spf} E(BG).$$

On the left, I've omitted the sp and viewed E_G as a covariant functor from spaces to schemes.

This approach to equivariant elliptic cohomology was suggested by Ginzburg-Kapranov-Vasserot and Grojnowski. In fact Grojnowski has constructed a contravariant functor

$$E_{S^1} : (S^1\text{-spaces}) \to (\text{sheaves of } \mathcal{O}_C\text{-algebras})$$

when C is the analytic elliptic curve $\mathbb{C}/(2\pi i\mathbb{Z} + 2\pi i\tau\mathbb{Z})$. It has the following properties.

1. A completion isomorphism
$$E_{S^1}(X)^{\wedge}_0 \cong E(X_{S^1}).$$

2. If X is a spin manifold, then $E_{S^1}(X^T)$ is an invertible $E_{S^1}(X)$-module, and there is a Pontryagin-Thom map
$$E_{S^1}(X^{-T}) \overset{\text{def}}{=} E_{S^1}(X^T)^{-1} \to E_{S^1}(*) = \mathcal{O}_C$$
which is compatible with the completion isomorphism.

Now consider the diagram

$$\begin{tikzcd}
\Gamma E_{S^1}(X^{V-T}) \arrow[r] \arrow[d] & E(V_{S^1} - T_{S^1}) \arrow[d] \\
\Gamma E_{S^1}(X^{-T}) \arrow[r] \arrow[d] & E(-T_{S^1}) \arrow[d] \\
\Gamma E_{S^1}(*) \arrow[r] & E(BS^1)
\end{tikzcd}$$

Now $E(V_{S^1} - T_{S^1})$ contains a class $W(M;V)$ which pushes forward to the Witten genus $w(M;V)_{S^1}$. On the other hand, the bottom left is just $\Gamma\mathcal{O}_C = \mathbb{C}$. Thus if we can construct a class $\gamma(M;V)$ in $\Gamma E_{S^1}(X^{V-T})$ which maps to $W(M;V)$ under the top horizontal map, i.e. such that $\gamma(M;V)_0$ is $W(M;V)$, then the Witten genus is rigid as required.

Theorem. *Let V and T be equivariant spin vector bundles of even rank over a S^1-space M. Suppose that $w_2(V-T)_{S^1}$ and $c_2(V-T)_{S^1} = 0$. Then S^1-orientations on V and T determine a trivialiation $\gamma(T;V)$ of $E_{S^1}(V) \otimes E_{S^1}(T)^{-1}$ as $E_{S^1}(X)$-module, such that $\gamma(T;V)_0 = W(T;V)$. Moreover the association $(T;V) \mapsto \gamma(T;V)$ is exponential under Whitney sum and natural under pull-back.*

(A S^1-orientation on V is a choice of orientation on the fixed subbundle V^A/M^A for each closed subgroup A of S^1; it turns out that any equivariant spin bundle is orientable).

Thus γ is a sort of analytic equivariant sigma orientation

$$MO\langle 8\rangle_{S^1} \to E_{S^1}.$$

I shall give a conceptual proof of the theorem which also sheds light on the nonequivariant sigma orientation. In order to do so it is useful to explain how theta functions arise in connection with degree-four characteristic classes.

c_2 and theta functions

Let G be a connected compact Lie group with maximal torus T and Weyl group W. Let $\widehat{T}$ and $\check{T}$ be as usual the lattices of characters and cocharacters of T. Then we have a map

$$\begin{aligned} H^4(BG;\mathbb{Z}) \to H^4(BT;\mathbb{Z})^W &\cong \mathsf{Sym}^2\,\widehat{T}^W \\ &\cong \mathsf{hom}(\Gamma_2\check{T},\mathbb{Z})^W. \end{aligned}$$

As I learned from Bill Dwyer, the arrow is an isomorphism if G is simply connected.

Thus a degree-four characteristic class c gives rise to a homomorphism $c : \Gamma_2\check{T} \to \mathbb{Z}$. Using c, define maps I and γ as follows.

$$\begin{tikzcd}
\check{T}\otimes\check{T} \arrow[r] \arrow[rr, bend left, "I"] & \Gamma_2\check{T} \arrow[r, "c"'] & \mathbb{Z} \\
& \check{T} \arrow[u, "\gamma_2"] \arrow[ur, "\phi"'] &
\end{tikzcd}$$

I is bilinear, and ϕ is quadratic; they are related by the formula

$$\phi(u+v) = \phi(u) + I(u,v) + \phi(v).$$

If M is an abelian group written multiplicatively and A is an abelian group written additively, then the rules for manipulation in $A \otimes M$ are most easily remembered if $a \otimes m$ is written m^a. With this convention, suppose that $0 < |q| < 1$ and $C = \mathbb{C}^\times / q^{\mathbb{Z}}$. Then

$$\check{T} \otimes C \cong (\check{T} \otimes \mathbb{C}^\times)/z \sim zq^a, a \in \check{T}.$$

Given ϕ and I defined by a degree-four characteristic class c, consider the line bundle over $\check{T} \otimes C$ given by the formula

$$\mathcal{L}(c) = \frac{\check{T} \otimes \mathbb{C}^\times \times \mathbb{C}}{(z, \lambda) \sim (zq^a, q^{\phi(a)} z^{\widehat{I}(a)} \lambda)}.$$

The W-invariance of c implies that $\mathcal{L}(c)$ is a W-equivariant line bundle over $\check{T} \otimes C$, and so it descends to a line bundle $\mathcal{A}(c)$ over $(\check{T} \otimes C)/W$. The sections of $\mathcal{A}(c)$ are precisely the W-invariant sections of $\mathcal{L}(c)$; these are holomorphic functions

$$\theta = \theta(z, q) : \check{T} \otimes \mathbb{C}^{\times} \to \mathbb{C}$$

such that

$$\begin{aligned} \theta(zq^{a}, q) &= q^{-\phi(a)} z^{-\widehat{I}(a)} \theta(z, q) \\ \theta(z^{w}, q) &= \theta(z, q) \end{aligned}$$

for $z \in \check{T} \otimes \mathbb{C}^{\times}$, $a \in \check{T}$, and $w \in W$.

These are precisely the equations which must be satsified by the character θ of a representation of the loop group LG of level c. In fact the Kac Character Formula identifies the $\mathbb{Z}((q))$-module of such theta functions (with integer cofficients) with the $\mathbb{Z}((q))$-span of characters of representations of LG of level c.

For example, consider the case that $G = Spin(2d)$. If we identify

$$\check{T}_{SO(2d)} \cong \mathbb{Z}^d$$

in the usual way, then

$$\check{T} \cong \{(m_1, \dots, m_d) | \sum m_i \equiv 0 \mod 2\}.$$

The class c_2 corresponds to

$$\phi(m) = \frac{1}{2} \sum m_i^2$$
$$I(m, m') = \sum m_i m'_i.$$

The "basic" representation of $LSpin(2d)$ is a representation of level c_2, with character

$$\sigma(u_1, \dots, u_d) = \prod_j \sigma(u_j, q).$$

Thus the product of sigma functions gives a section of $\mathcal{A}(c_2)$. Of course it is not a trivialization, but if $I(\sigma)$ is the ideal sheaf defined by the zeroes of χ, then σ does give a trivialization of $\mathcal{A}(c_2) \otimes I(\sigma)$.

In the course of proving the Theorem, I make a series of conjectures which amount to the following (see math.AT/0201092).

Let $B_{S^1}G$ be the classifying space for S^1-equivariant principal G-bundles. First of all, there should be a canonical isomorphism

$$E_{S^1}(B_{S^1}G) \cong (\check{T} \otimes C)/W.$$

Thus if V is a S^1-equivariant $G = Spin(2d)$-vector bundle over X, the map

$$X \to B_{S^1}G$$

classifying V gives rise to a map

$$E_{S^1}(X) \to (\check{T} \otimes C)/W.$$

By pulling back the line bundles $\mathcal{A}(c_2)$ and $I(\sigma)$ and the section σ, we obtain line bundles $\mathcal{A}(V)$ and $I(V)$ over $E_{S^1}(X)$, and a trivialization $\sigma(V)$ of $\mathcal{A}(V) \otimes I(V)$.

I further conjecture that there is a canonical isomorphism

$$I(V) \cong E_T(X^V)$$

of $E_T(X)$-modules, and that if V' is another equivariant spin bundle such that

$$c_2(V - V') = 0,$$

then a $BO\langle 8\rangle_{S^1}$-structure on $V - V'$ determines a trivialization of $\mathcal{A}(V) \otimes \mathcal{A}(V)^{-1}$. In the presence of such a trivialization, then, $\sigma(V)/\sigma(V')$ is a trivialization of

$$\frac{\mathcal{A}(V) \otimes I(V)}{\mathcal{A}(V') \otimes I(V')} \cong \frac{I(V)}{I(V')} \cong \frac{E_{S^1}(X^V)}{E_{S^1}(X^{V'})}.$$

This is the sigma orientation; the theorem was proved by writing down the formulae implied by these conjectures.