

1. SOME REASONS THAT I CARE ABOUT p -DIVISIBLE GROUPS.

I am only a dabbler in the world of p -divisible groups so you should probably take anything I say with a grain of salt unless you happen to see me actually prove it.

1.1. Understanding their moduli. Let h be a positive integer and let \mathcal{M} be the moduli stack, in the étale topology, of one-dimensional p -divisible groups of height h over $\mathrm{Spec} \overline{\mathbf{F}}_p$. I think I understand this stack reasonably well and in fact I can sketch a method for computing a Hopf algebroid whose associated stack is this stack.

All of this goes out the window in characteristic zero, however: let \mathcal{M}' be the moduli stack, in the étale topology, of one-dimensional p -divisible groups of height h over $\mathrm{Spf} W(\overline{\mathbf{F}}_p)$. I don't claim to understand this stack at all, or even what kind of world you have to imagine it living in in order for it to be geometric (i.e. admitting an affine cover by an affine formal scheme). Here is one way in which \mathcal{M}' is mystifying to me: the simpler stack \mathcal{M} has h geometric points, because each p -divisible group classified by \mathcal{M} is isomorphic to one on the following list:

$$G_0^{h-1} \times G_1, G_0^{h-2} \times G_{1/2}, \dots, G_0 \times G_{1/(h-1)}, G_{1/h}.$$

But for each one of these points, say, the one classifying p -divisible groups isomorphic to a particular p -divisible group G on the above list, \mathcal{M}' should have a closed point sitting inside of an open substack such that completion along the inclusion of the closed point yields the Rapoport-Zink space classifying deformations of G . Rapoport-Zink spaces have some bad properties, though; in particular they rarely have ideal of definition (p) . For example, height h Lubin-Tate space, which is the Rapoport-Zink space classifying deformations of $G_{1/h}$, is isomorphic to $\mathrm{Spf} W(\overline{\mathbf{F}}_p)[[u_1, \dots, u_{h-1}]]$, so it has ideal of definition (p, u_1, \dots, u_{h-1}) , not (p) . Understanding the ideal of definition of a Rapoport-Zink space is not straightforward and in fact there is an open conjecture about which Rapoport-Zink spaces have ideal of definition (p) . Even worse, Rapoport-Zink spaces sometimes fail to be flat over their base scheme.

Compare this to what happens with the moduli stack \mathcal{M}_{fg} of 1-dimensional formal groups (of arbitrary, not-necessarily-constant height), over $\mathrm{Spf} W(\mathbf{F}_p)$: while Lubin-Tate spaces (which are completions of inclusions of closed points into open substacks of this moduli stack, as Paul Goerss showed) have ideals of definition that are bigger than we would like, \mathcal{M}_{fg} is essentially built up out of these Lubin-Tate spaces, and \mathcal{M}_{fg} is a very well-behaved stack: as a stack (in the fpqc topology) fibered in groupoids over formal schemes over $\mathrm{Spf} W(\mathbf{F}_p)$ with ideal of definition (p) , it admits an affine cover by $\mathrm{Spf} \hat{\mathbf{Z}}_p[v_1, v_2, \dots]$; so in that world it's geometric (and in fact it's the stack associated to the Hopf algebroid for the p -completed BP -spectrum). So the messy ideals of definition of Lubin-Tate spaces don't cause any trouble, and \mathcal{M}_{fg} is still a geometric stack fibered in groupoids over formal schemes with a very nice ideal of definition. The question is whether a similar thing happens with \mathcal{M}' , which is, in a particular sense, built from (possibly messy) Rapoport-Zink spaces.

In particular, \mathcal{M} has a stratification by closed substacks, in which the i th closed substack classifies p -divisible groups of formal height $\geq i$. Does this stratification lift to \mathcal{M}' ? In the rest of this document I'll assume it does.

1.2. Differentials in the chromatic spectral sequence. (This is what Mark Behrens calls “trans-chromatic phenomena.”)

A consequence of a theorem of Jacob Lurie’s is that, to any Deligne-Mumford stack \mathcal{X} (possibly fibered in groupoids over formal schemes with a particular ideal of definition) equipped with a sufficiently nice map $\mathcal{X} \rightarrow \mathcal{M}'$, there is canonically associated a (homotopy) sheaf \mathcal{E} of E_∞ -ring spectra such that $\pi_0 \circ \mathcal{E}$ recovers the structure ring sheaf of the local étale site on \mathcal{X} . Suppose $\mathcal{X} \rightarrow \mathcal{M}'$ satisfies these conditions, and suppose U is an étale open of \mathcal{X} whose image in \mathcal{M}' intersects exactly i strata, for some positive integer $i \leq h$. Then we have the chromatic tower

$$\begin{array}{ccc}
 & \cdots & \\
 & \downarrow & \\
 \mu_3 \Gamma(\mathcal{E}, U) & \longrightarrow & L_3 \Gamma(\mathcal{E}, U) \\
 & & \downarrow \\
 \mu_2 \Gamma(\mathcal{E}, U) & \longrightarrow & L_2 \Gamma(\mathcal{E}, U) \\
 & & \downarrow \\
 \mu_1 \Gamma(\mathcal{E}, U) & \longrightarrow & L_1 \Gamma(\mathcal{E}, U) \\
 & & \downarrow \\
 & & L_0 \Gamma(\mathcal{E}, U)
 \end{array}$$

with the homotopy fibers of the projections written to the left of the tower. The chromatic spectral sequence for $\Gamma(\mathcal{E}, U)$ then takes the form

$$\begin{aligned}
 E_1^{*,*,*} &\cong \text{Cotor}_{BP_*BP}^*(BP_*\mu_*\Gamma(\mathcal{E}, U), BP_*) \\
 &\Rightarrow \text{Cotor}_{BP_*BP}^*(BP_*\Gamma(\mathcal{E}, U), BP_*).
 \end{aligned}$$

Is the maximal length of a differential in the chromatic spectral sequence for $\Gamma(\mathcal{E}, U)$ exactly $i - 1$? If U is an étale neighborhood of a curve in \mathcal{M}' which meets the codimension 2 stratum orthogonally to the codimension 1 stratum, for example, does the chromatic spectral sequence for $\Gamma(\mathcal{E}, U)$ have d_2 differentials but no d_1 differentials?

On the other hand, if we have a differential in the chromatic spectral sequence for the sphere spectrum, can we detect it with a map to the chromatic spectral sequence for $\Gamma(\mathcal{E}, U)$ for some carefully chosen \mathcal{E} (and height h)?

1.3. Milnor τ s, étale quotients, and algebraic Novikov spectral sequences.

Let $p > 2$. Then the $H\mathbf{F}_p$ -Adams spectral sequence (for the sphere), which from now I will call the classical Adams spectral sequence, takes the form

$$\text{Cotor}_{(H\mathbf{F}_p)_*H\mathbf{F}_p}^*((H\mathbf{F}_p)_*, (H\mathbf{F}_p)_*) \Rightarrow \pi_*(S^0) \otimes \check{Z}_p$$

while the BP -Adams spectral sequence, which from now on I will call the ANSS, takes the form

$$\text{Cotor}_{BP_*BP}^*(BP_*, BP_*) \Rightarrow \pi_*(S^0) \otimes Z_{(p)}.$$

The main difference is that the classical ASS has more differentials than the ANSS, despite converging to essentially the same thing; there is more “noise” in the classical Adams E_2 which has to eventually be killed by differentials. The culprit here

is the Milnor τ elements:

$$\begin{aligned} (H\mathbf{F}_p)_*H\mathbf{F}_p &\cong P(\xi_1, \xi_2, \dots) \otimes E(\tau_0, \tau_1, \dots) \\ (H\mathbf{F}_p)_*BP &\cong P(\xi_1, \xi_2, \dots). \end{aligned}$$

In some sense, spectra E whose mod p homology admit nontrivial coaction by Milnor τ s have “noise” in their E -Adams spectral sequence which has to be killed by differentials.

This is made more precise by the algebraic Novikov spectral sequence: this is a spectral sequence whose E_1 -term is (a regrading of) the E_2 -term of the classical ASS, and which converges to the E_2 -term of the ANSS. The algebraic Novikov spectral sequence “takes care of” (in the mafia sense) precisely the “noise” in the Adams spectral sequence introduced by the Milnor τ s.

I would like to know if, to any ring spectrum $\Gamma(\mathcal{E}, U)$ (with U affine) coming from an application of Lurie’s theorem as above, there exists some canonically associated complex oriented ring spectrum X with a map $X \xrightarrow{\eta} \Gamma(\mathcal{E}, U)$ with the following properties (or at least properties close to these):

- For any particular choice of coordinate, the map $\pi_0 MUP \rightarrow \pi_0 \Gamma(\mathcal{E}, U)$ classifying the underlying formal group law of the p -divisible group on $\text{Spec } \pi_0 \Gamma(\mathcal{E}, U) \cong U$ factors as the composite of the complex orientation map $\pi_0 MUP \rightarrow \pi_0 X$ with the map

$$\pi_0 X \xrightarrow{\pi_0(\eta)} \pi_0 \Gamma(\mathcal{E}, U).$$

- The map $X \rightarrow \Gamma(\mathcal{E}, U)$ induces a homotopy equivalence

$$(S^0)_{\check{X}} \xrightarrow{\simeq} (S^0)_{\check{\Gamma(\mathcal{E}, U)}},$$

i.e., the homotopy limits of the X -Adams and $\Gamma(\mathcal{E}, U)$ -Adams towers coincide.

- The Milnor τ elements in the dual Steenrod algebra coact trivially on $H_*(X; \mathbf{F}_p)$.
- There exists a spectral sequence, analogous to the algebraic Novikov spectral sequence, with

$$\begin{aligned} E_1^{*,*} &\cong \text{Cotor}_{\Gamma(\mathcal{E}, U)_* \Gamma(\mathcal{E}, U)}^*(\Gamma(\mathcal{E}, U)_*, \Gamma(\mathcal{E}, U)_*) \\ &\Rightarrow \text{Cotor}_{X_* X}^*(X_*, X_*). \end{aligned}$$

The idea is that X is like “the formal component of $\Gamma(\mathcal{E}, U)$ ” and its relation to $\Gamma(\mathcal{E}, U)$ is very much like the relation of BP to $H\mathbf{F}_p$; if this stuff is true then this implies a fairly close and precise relationship between the étale quotients of p -divisible groups, Milnor τ s, and differentials in generalized Adams spectral sequence.

1.4. “Chromatic base change.” If K/\mathbf{Q}_p is a finite extension then there is a map from the moduli stack of formal \mathcal{O}_K -modules to the moduli stack of formal groups (over $\text{Spec } \mathbf{Z}_{(p)}$), and the map this induces in fpqc cohomology is very useful for computations; in particular, the differentials in the May spectral sequences converging to the cohomology of the height h Morava stabilizer group can be computed in terms of restrictions (in group cohomology) induced on the level of stabilizer groups by the above map of moduli stacks, for K/\mathbf{Q}_p totally ramified and of degree dividing h . I think of this as “chromatic base change.”

There is an analogous picture for p -divisible groups: if K/\mathbf{Q}_p is a finite extension then a p -divisible group G equipped with an appropriate injection $\mathcal{O}_K \hookrightarrow \text{End}(G)$

is called a “Barsotti-Tate \mathcal{O}_K -module.” There is a theory of crystals and isocrystals for such things which supposedly (according to the book of Rapoport and Zink) parallels that of p -divisible groups and their crystals and isocrystals, but I have never seen this written down anywhere (although it is supposed to be implicit in Drinfeld’s “Coverings of p -adic symmetric domains”). I would like to see this theory worked out explicitly. In particular, in my thesis I worked out some families of splittings of the moduli stacks of formal \mathcal{O}_K -modules, and I would be interested in knowing whether moduli stacks of Barsotti-Tate \mathcal{O}_K -modules admit similar splittings.