A Few Review Problems for Chapters 5 & 6

Math 225

December 9, 2008

1. Let A be the matrix below. Which of 1, 2, -1 are eigenvalues for A? Which of $(1, 1, 1, 1)^T$, $(2, 1, 2, 1)^T$, $(0, 0, 0, 0)^T$ are eigenvectors for A?

$$A = \begin{vmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{vmatrix}$$

   2 is an eigenvalue, $(1, 1, 1, 1)^T$ is an eigenvector.

2. Suppose that A is a 3×3 matrix whose eigenvalues are -7, 1, 2. Find, if possible, eigenvalues of $A + 5I_3$.

   $\det(A - \lambda I)$=0 if and only if $\lambda = -7,\ 1,\ 2.$
   thus $\det(A - (\lambda - 5)I)$=0 if and only if $\lambda - 5 = -7,\ 1,\ 2.$
   so $\lambda = -2,\ 6,\ 7$ are the eigenvalues for $A + 5I$.

3. Let $A^2 = 0$, where A is an $n \times n$ matrix. Show that if λ is an eigenvalue of A, then $\lambda = 0$.

   If $A\mathbf{x} = \lambda\mathbf{x}$ for some $\mathbf{x} \neq 0$ then $A^2\mathbf{x}=A(A\mathbf{x})=A(\lambda\mathbf{x})=\lambda A\mathbf{x} = \lambda^2\mathbf{x}$.
   But $A^2\mathbf{x}$=0 since $A^2 = 0$. So $\lambda^2\mathbf{x}$=0.
   But $\mathbf{x} \neq 0$, so $\lambda = 0$.

4. Diagonalize A, if possible. That is, find matrices P and D (diagonal) so that $A = PDP^{-1}$.

$$A_1 = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad A_4 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

   A_1 and A_2 are diagonalizable. An example of each diagonalization is below. A_3 and A_4 are not diagonalizable, since the eigenspaces corresponding to $\lambda = 0$ have dimension 1, not 2.

$$P_1 = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}; \quad P_2 = \begin{bmatrix} -2 & 1 & 0 \\ -1 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

5. Find the orthogonal projection of $(1, 2, 2, 2)^T$ onto $(3, 2, 1, 2)^T$.

   $\hat{\mathbf{y}} = \frac{13}{18}(3, 2, 1, 2)^T$.

6. Determine if the set of vectors is orthonormal. If the set is only orthogonal, normalize the vectors to produce an orthonormal set.

$$u_1 = \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix}$$

   The set is orthogonal since $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$, $\mathbf{u}_1 \cdot \mathbf{u}_3 = 0$, and $\mathbf{u}_2 \cdot \mathbf{u}_3 = 0$.
   The set is not orthonormal since even though $\mathbf{u}_1 \cdot \mathbf{u}_1 = 0$, we have $\mathbf{u}_2 \cdot \mathbf{u}_2 \neq 0$. So we must normalize $\mathbf{u}_2$. Let $\mathbf{y} = 3\mathbf{u}_2$. Then $||\mathbf{y}|| = \sqrt{5}$. So let the new $\mathbf{u}_2$ be $\frac{y}{||y||} y = (\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0)^T$.

7. Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$. Then express $\mathbf{x}$ as a linear combination of the $\mathbf{u}$'s.

$$u_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \qquad x = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$$

   The set is orthogonal since $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$, $\mathbf{u}_1 \cdot \mathbf{u}_3 = 0$, and $\mathbf{u}_2 \cdot \mathbf{u}_3 = 0$. Thus, by theorem, the set is linearly independent, and hence a basis for $\mathbb{R}^3$ (since 3 vectors). Then, recall $\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$ where $c_i = \frac{\mathbf{x} \cdot \mathbf{u}_i}{\mathbf{u}_i \cdot \mathbf{u}_i}$. So:

$$\mathbf{x} = \frac{4}{3}\mathbf{u}_1 + \frac{1}{3}\mathbf{u}_2 + \frac{1}{3}\mathbf{u}_3$$

8. Write $\mathbf{y}$ as the sum of two orthogonal vectors, one in Span$\{\mathbf{u}\}$ and one orthogonal to $\mathbf{u}$.

$$\mathbf{y} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} = .3\mathbf{u} = (1.8, \ .6)^T$$

which is in span$\{\mathbf{u}\}$.

$$\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = (-.8,\ 2.4)^T$$

which is orthogonal to $\mathbf{u}$ So $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z} = (1.8,\ .6)^T + (-.8,\ 2.4)^T$.