Math 234: 9.2 - Integration by Parts

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Integration by parts is a technique of integration based on the product rule for differentiation. Let $u(x)$ and $v(x)$ be differentiable functions of x. Then consider the product rule:

$$\frac{d}{dx}[u(x)v(x)] = u(x)\frac{dv}{dx} + v(x)\frac{du}{dx}$$

we rearrange this formula as:

$$u(x)\frac{dv}{dx} = \frac{d}{dx}[u(x)v(x)] - v(x)\frac{du}{dx}$$

Integrating both sides of the equation with respect to x gives,

$$\int [u(x)\frac{dv}{dx}]dx = u(x)v(x) - \int [v(x)\frac{du}{dx}]dx$$

We write this formula more compactly below. This is called the integration by parts formula.

$$\int u\ dv = uv - \int v\ du$$

Integration by parts is a very important method of integration. Suppose you have $\int f(x)dx$. If you can write it as $\int u\ dv$, then

$$\int f(x)dx = uv - \int v\ du$$

If $\int v\ du$ is easier to compute than your original integral, then this method has helped you to evaluate your integral!

Let's see this formula in action:

EXAMPLE 1: Compute $\int xe^{2x}dx$

Since x becomes simpler when differentiated we will choose

$$u = x, \qquad dv = e^{2x}dx$$

So we compute

$$du = dx, \qquad v = \int e^{2x}dx = (1/2)e^{2x}$$

We will leave out the "+C" until the end, for simplicity.
Then, using our formula we have

$$\int xe^{2x}dx = x(\frac{1}{2}e^{2x}) - \int \frac{1}{2}e^{2x}dx$$

$$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$$

OUR GENERAL STRATEGY:

1. Choose functions u and dv so that $f(x)dx = u\ dv$. Try to pick u so that du is simpler than u and a dv that is easy to integrate.

2. Compute $u = ...$, $du = ...$, $dv = ...$, and $v = \int dv = ...$

3. Complete the integration by finding $uv - \int v\ du$. Then

$$\int f(x)dx = \int u\ dv = uv - \int v\ du$$

Add "+C" at the end of the computation!

EXAMPLE 2: Find $\int x^2 \ln x \; dx$

$$u = \ln x \qquad dv = x^2 dx$$
$$du = \qquad\qquad v =$$

EXAMPLE 3: Find $\int x\sqrt{x+5} \; dx$

$$u = \qquad\qquad dv =$$
$$du = \qquad\qquad v =$$

NOTE: It is also possible to do this problem by using the substitution $u = x + 5$, $du = dx$, $x = u - 5$.
Then $\int x\sqrt{x+5} \; dx = \int (u-5)\sqrt{u} \; du$

$$= \int (u^{3/2} - 5u^{1/2}) du = \frac{2}{5}u^{5/2} - \frac{10}{3}u^{3/2} + C$$

$$= \frac{2}{5}(x+5)^{5/2} - \frac{10}{3}(x+5)^{3/2} + C$$

You might notice that the two solutions result in answers that look different. To see that they are the same, consider the following computation:

$$\frac{2}{5}(x+5)^{5/2} - \frac{10}{3}(x+5)^{3/2} = (x+5)^{3/2}(\frac{2x+10}{5} - \frac{10}{3})$$
$$= (x+5)^{3/2}(\frac{6x+30-50}{15}) = (x+5)^{3/2}(\frac{2}{3}x - \frac{4}{15}x - \frac{20}{15})$$
$$= \frac{2x}{3}(x+5)^{3/2} - \frac{4}{15}(x+5)^{5/2}$$

Sometimes we may need to apply the formula twice! Consider the following example:

EXAMPLE 3: Find $\int x^2 e^{3x} dx$

$$u = \qquad\qquad dv =$$
$$du = \qquad\qquad v =$$

What happens if your integrand has only one "part"?

EXAMPLE 4: Find $\int \ln x \; dx$

$$u = \qquad\qquad dv =$$

$$du = \qquad\qquad v =$$

HOW DO I KNOW WHICH TECHNIQUE TO USE?

When you look at the integrand, can you see a function and it's derivative? If yes, then you should try integration by substitution first! Let's try some problems to get an idea of this process:

EXAMPLE 5: Find $\int \frac{(\ln x)^5}{x} dx$

EXAMPLE 6: Find $\int \frac{\ln x}{x^5} dx$

EXAMPLE 7: Find $\int \frac{2x}{(x-2)^3} dx$

EXAMPLE 8: Find $\int \frac{2x}{(x^2-2)^3} dx$