



**SOLUTIONS: Review Problems for Exam 1:
1.09, 2.05, VC.01-VC.06**

Exam 1: 10/21/2009

Mathematica 6.0 Initializations

□ R.1)

Go with the ellipse $\left(\frac{x-1}{4}\right)^2 + \left(\frac{y+2}{3}\right)^2 = 1$

Write down a parametric formula for the ellipse where t advances from 0 to 2π .

$$\{\mathbf{x}[t], \mathbf{y}[t]\} = \{1, -2\} + \{4 \cos[t], 3 \sin[t]\}$$

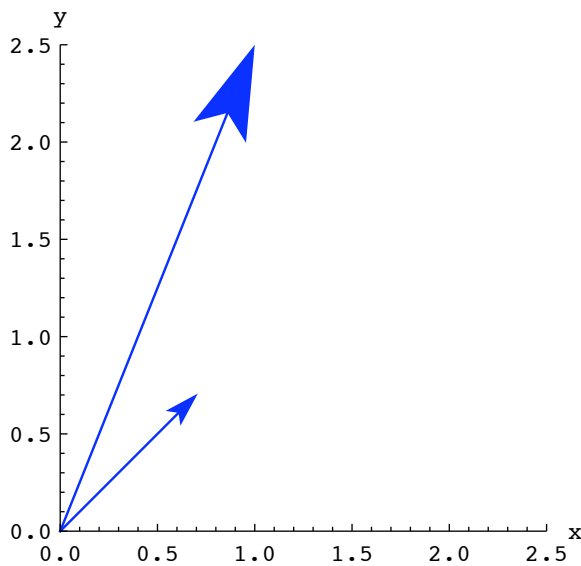
□ R.2)

Give the number t that makes $X = \{3, -1, 2\}$ and $Y = \{1, 1, t\}$ perpendicular.

$$t = -1, \text{ since this will make } \mathbf{X} \cdot \mathbf{Y} = 0.$$

□ R.3a)

Here are the vectors $X = \left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}$ and $Y = \{1, 2.5\}$ with their tails at $\{0, 0\}$, shown in true scale:



Identify X and Y , and pencil in the component of Y in the direction of X . Calculate the component of Y in the direction of X .

Y is the longer vector in the picture above.

$$\mathbf{Y}_{\text{push along } \mathbf{X}} = \mathbf{X} (\mathbf{X} \cdot \mathbf{Y}) / (\mathbf{X} \cdot \mathbf{X}) = \{1.75, 1.75\}$$

If I could draw on this *Mathematica* document, then I would extend X out until a line perpendicular to X would go through the tip of Y . The push of Y in the direction of X is the multiple of X whose tip is closest to the tip of Y when you put the tails of both vectors at $\{0, 0\}$

□ R.3b)

If $X \cdot Y > 0$, then is the push of X in the direction of Y with Y or against Y?

If $X \cdot Y < 0$, then is the push of X in the direction of Y with Y or against Y?

If $X \cdot Y > 0$, then the push of X in the direction of Y is WITH Y.

If $X \cdot Y < 0$, then the push of X in the direction of Y is AGAINST Y.

□ R.4)

Write a parametric formula for the line that passes through the points $\{3, 1\}$ and $\{5, 11\}$.

Give a vector parallel to this line.

Give a vector perpendicular to this line.

$\{x[t], y[t]\} = \{3, 1\} + t\{2, 10\}$

A vector parallel to this line is $\{2, 10\}$

A vector perpendicular to this line is $\{-10, 2\}$

□ R.5)

Given $X = \{1, 1, 1\}$ and $Y = \{2, 0, 0\}$, calculate $X \cdot Y$ and $X \times Y$.

$X \cdot Y = 2$

$X \times Y = \{0, 2, -2\}$

□ R.6)

Given that X and Y are perpendicular unit vectors in 2D, describe the curve traced out by

$P[t] = \text{Cos}[t] X + \text{Sin}[t] Y$ as t runs from 0 to 2π .

This is the unit circle, centered at the origin.

□ R.7)

You are walking around a closed curve with no loops (a curve like a distorted circle) in the counterclockwise way, and at time t, you are at the point $\{x[t], y[t]\}$.

Does the unit normal vector

$$\frac{\{y'[t], -x'[t]\}}{\sqrt{x'[t]^2 + y'[t]^2}}$$

transplanted with its tail at $\{x[t], y[t]\}$ point out away from the curve toward your right foot, or does it point inside the curve toward your left foot?

It points away from the curve towards my right foot.

□ R.8)

Calculate the gradient, $\nabla f[x, y] = \text{grad}f[x, y]$, of

$$f[x, y] = e^{-3xy}.$$

Calculate the gradient, $\nabla f[x, y, z] = \text{grad}f[x, y, z]$, of

$$f[x, y, z] = x \text{Sin}[xyz].$$

$$\text{grad}f[x, y] = \{-3ye^{-3xy}, -3xe^{-3xy}\}$$

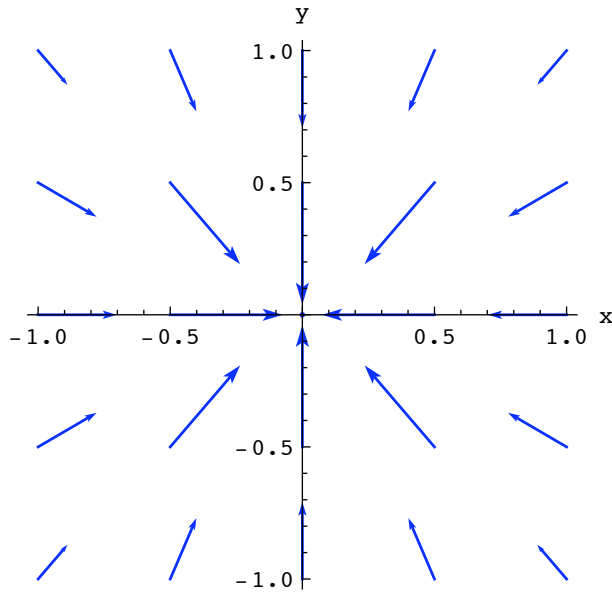
$$\text{grad}f[x, y, z] = \{\text{Sin}[xyz]+xyz\text{Cos}[xyz], x^2 z\text{Cos}[xyz], x^2 y\text{Cos}[xyz]\}$$

□ R.9)

Go with

$$f[x, y] = \frac{0.6}{1 + 1.2x^2 + 1.4y^2},$$

and look at a plot of $\text{grad}f[x, y]$ with tails at $\{x, y\}$ for some selected points $\{x, y\}$ in the vicinity of $\{0, 0\}$:



Why are these gradient vectors so attracted to the point $\{0, 0\}$?

$\{0,0\}$ minimizes the denominator, and thus maximizes $f[x,y]$. The gradient vectors point in the direction of greatest initial increase, thus the vectors in the gradient field point towards the max at $\{0,0\}$.

□ R.10)

Go with

$$f[x, y, z] = x y \sin[z].$$

In which direction should you leave the point $\{1, 2, 0\}$ to get the greatest possible initial increase of $f[x, y, z]$?

In which direction should you leave the point $\{1, 2, 0\}$ to get the greatest possible initial decrease of $f[x, y, z]$?

$$\text{grad}f[x, y, z] = \{y\sin[z], x\sin[z], xy\cos[z]\}$$

$\text{grad}f[1,2,0] = \{0,0,2\}$, thus we should leave $\{1,2,0\}$ in the direction $\{0,0,2\}$ to get the greatest possible initial increase of $f[x,y,z]$.

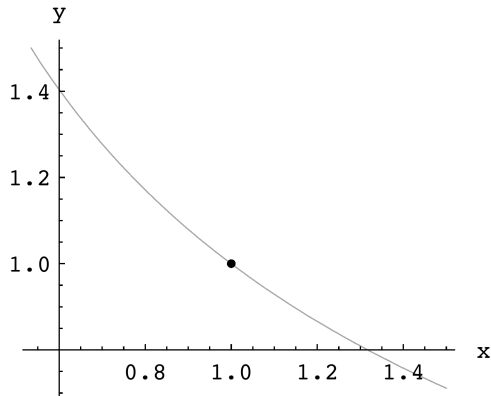
We should leave $\{1,2,0\}$ in the direction $\{0,0,-2\}$ to get the greatest possible initial decrease of $f[x,y,z]$.

□ R.11)

The point $\{1, 1\}$ is on the level curve

$$x^2 y^3 e^{x+y} = 7.38906:$$

Here's a plot of part of this level curve with axes going through the point $\{1, 1\}$:



Pencil in a vector with its tail at $\{1, 1\}$ that points in the same direction as $\text{grad}f[1, 1]$.

$$\text{grad}f[x, y] = \{(2x y^3 + x^2 y^3)e^{x+y}, (3x^2 y^2 + x^2 y^3)e^{x+y}\}$$

Thus, $\text{grad}f[1,1]$ has both entries positive (although you were probably able to tell this without calculating $\text{grad}f$ out). The gradient vector will be perpendicular to the curve (as they are always perpendicular to level curves) and will point up and to the right. Any vector meeting this description is fine here.

□ R.12)

Does

$$f[x, y] = x^6 - 3x^2y + y^4 - 6x + 5y$$

have a maximizer or minimizer? How do you know?

When $|x|$ and $|y|$ are large then the dominant terms

$$x^6 + y^4$$

make $f[x, y]$ really huge, so the global scale plot of $f[x, y]$ looks like a cup. This means $f[x, y]$ has no tallest crest but does have a deepest dip. As a result, $f[x, y]$ has no maximum value but does have a minimum value.

Does

$$f[x, y] = \frac{x^5 + 3y^2}{1 + x^4 + y^6}$$

have a maximizer or a minimizer? How do you know?

Also $f[x, 0] > 0$ for $x > 0$ and $f[x, 0] < 0$ for $x < 0$.

Since for large positive x 's $f[x,0]$ runs off to infinity, and for large negative x 's, $f[x,0]$ runs off to minus infinity, there is neither a maximizer nor a minimizer.

R.13)

Calculate

$$\frac{\partial f[x,y,z]}{\partial z} = f^{(0,0,1)}[x, y, z]$$

by hand for

$$f[x, y, z] = \text{Sin}[x^2 y^3 z^4].$$

$$\frac{\partial f[x,y,z]}{\partial z} = f^{(0,0,1)}[x, y, z] = 4x^2 y^3 z^3 \text{Cos}[x^2 y^3 z^4]$$

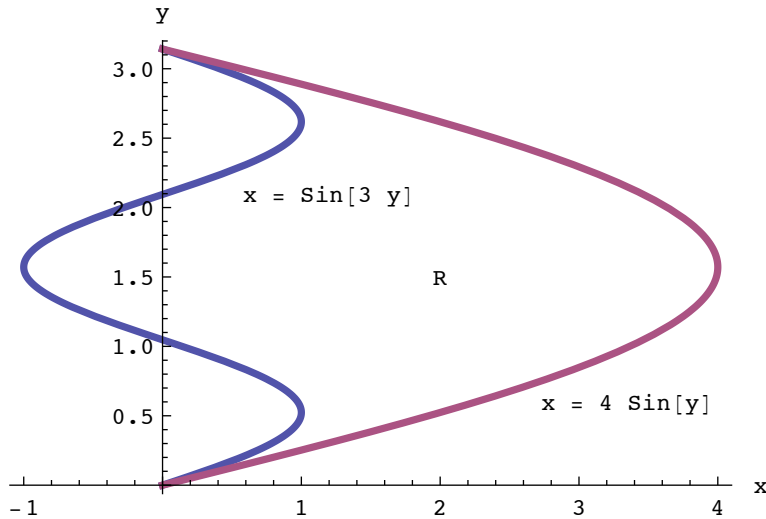
□ R.14)

Here is a 2D region R whose left boundary is the curve

$$\text{left}[y] = \text{Sin}[3 y]$$

and whose right boundary is the curve

$$\text{right}[y] = 4 \text{Sin}[y]:$$



Say why it's natural to calculate

$$\iint_{\mathcal{R}} 1 \, dx \, dy$$

by integrating with respect to x first.

Next, calculate

$$\iint_{\mathcal{R}} 1 \, dx \, dy$$

by hand and say what

$$\iint_{\mathcal{R}} 1 \, dx \, dy$$

measures.

The integral measures the area of the enclosed region. It is natural to integrate first with respect to x since it is natural to cut the region with horizontal slices. This would be the simplest way of slicing the above region.

$$= \int_0^{\pi} \int_{\sin[3y]}^{4 \sin[y]} 1 \, dx \, dy = \int_0^{\pi} 4 \sin[y] - \sin[3y] \, dy = 8 - (2/3) = 22/3$$

□ R.15)

Go with

$$f[x, y] = 2x + 5 \sin[y].$$

Give a clean formula for the function $n[x, y]$ defined by

$$n[x, y] = \int_0^x f[s, y] \, ds.$$

How are $f[x, y]$ and $D[n[x, y], x]$ related?

Is the outcome an accident? Why or why not?

$$n[x,y] = \int_0^x f[s, y] ds = x^2 + 5x\sin[y] + C.$$

We can see that $f[x,y] = 2x + 5\sin[y] = D[n[x,y], x]$.

This is not an accident, it is always true if we define $n[x,y]$ in this way.

We are quite happy that this is true so that we can apply the Gauss-Green formula!

□ R.16)

You are given that the point $\{2, 1\}$ is on a certain trajectory in the vector field

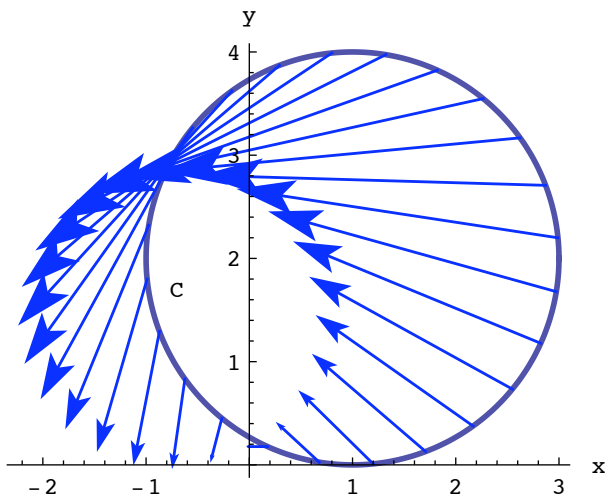
$$\text{Field}[x, y] = \{x + y, x - y\}.$$

Write down a vector that is tangent to this trajectory at the point $\{2, 1\}$.

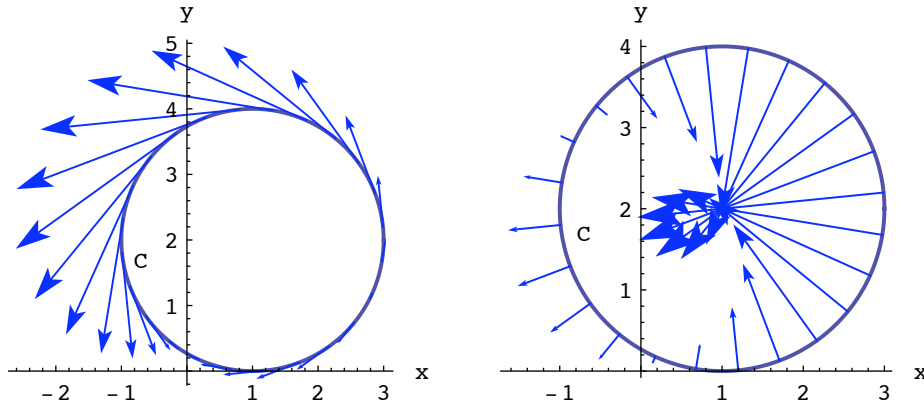
$\{3, 1\}$ is tangent to the trajectory at $\{2, 1\}$

□ R.17)

Here's a circle and the plot of a certain vector field $\text{Field}[x, y]$ plotted with tails at $\{x, y\}$ for a generous selection of points $\{x, y\}$ on the circle.



Here are tangential and normal components of what you see above:



Look at these plots, and then estimate whether the net flow of this vector field across the circle is from inside to outside, or is from outside to inside, and estimate whether the net flow of this vector field along this circle is clockwise or counterclockwise.

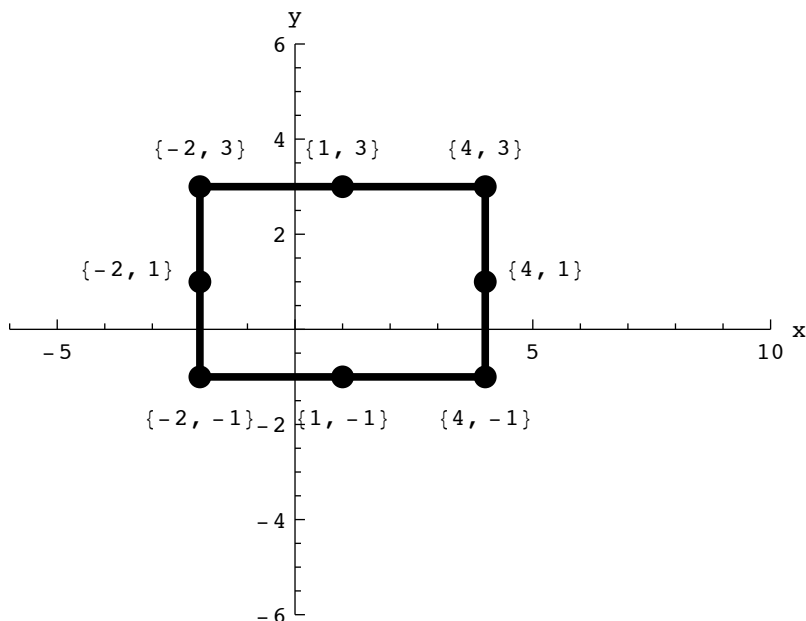
The net flow of this vector field across the circle is from outside to inside.

The net flow of this vector field along the circle is counterclockwise.

□ R.18)

Here is a rectangle C with eight labeled points:

|



Go with the vector field

$$\text{Field}[x, y] = \{y + x, -x\}.$$

For each labeled point $\{x, y\}$, pencil in the field vector $\text{Field}[x, y]$ with tail at $\{x, y\}$.

On the basis of your plot, estimate whether the net flow of this vector field across the edge of this rectangle is from inside to outside, or is from outside to inside, and estimate whether the net flow of this vector field along this curve is clockwise or counterclockwise.

Since most of my arrows point from inside to outside, I'd say the net flow of the vector field across the edge of the rectangle is from inside to outside. Similarly, I'd say that the net flow along the curve is clockwise.

□ R.19)

Suppose $f[x_0, y_0] > f[x, y]$ for all other $\{x, y\}$ near $\{x_0, y_0\}$. If you center a small circle C at $\{x_0, y_0\}$, why do you expect that the net flow of $\text{grad}f[x, y]$ across C is from outside to inside?

Suppose $f[x_0, y_0] < f[x, y]$ for all other $\{x, y\}$ near $\{x_0, y_0\}$. If you center a small circle C at $\{x_0, y_0\}$, why do you expect that the net flow of $\text{grad}f[x, y]$ across C is from inside to outside?

In the first case, I expect the net flow to be from outside to inside since gradient vectors point in the direction of greatest initial increase - which means they will point generally towards the max.

In the second case I expect the net flow to be from inside to outside for the same reason.

□ R.20)

You have a given vector field

$$\text{Field}[x, y] = \{m[x, y], n[x, y]\}.$$

You have parameterized the ellipse

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

in the counterclockwise way with

$$\{x[t], y[t]\} = \{2 \text{Cos}[t], 3 \text{Sin}[t]\}$$

with $0 \leq t \leq 2\pi$.

You calculate

$$\begin{aligned} & \oint_C m[x, y] dx + n[x, y] dy \\ &= \int_0^{2\pi} (m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t]) dt, \end{aligned}$$

and learn that

$$\oint_C m[x, y] dx + n[x, y] dy = 9.01.$$

Then you calculate

$$\begin{aligned} & \oint_C -n[x, y] dx + m[x, y] dy \\ &= \int_0^{2\pi} (-n[x[t], y[t]] x'[t] + m[x[t], y[t]] y'[t]) dt, \end{aligned}$$

and learn that

$$\oint_C -n[x, y] dx + m[x, y] dy = -3.52.$$

Is the net flow of $\text{Field}[x, y]$ along C clockwise or counterclockwise?

Is the net flow of $\text{Field}[x, y]$ across C from outside to inside, or is it from inside to outside?

The net flow along C is counterclockwise since the integral was positive.

The net flow across C means from outside to inside since the integral was negative.

CAUTION: We knew we had a counterclockwise parameterization here!

□ R.21)

Measure by hand calculation the net flow across the circle

$$x^2 + y^2 = 4$$

of a fluid whose velocity is given by the vector field

$$\text{Field}[x, y] = \{x, y\}.$$

Don't forget that $\text{Sin}[t]^2 + \text{Cos}[t]^2 = 1$.

Is the net flow of $F[x, y]$ across this curve from inside to outside or outside to inside?

For $\{m[x,y], n[x,y]\} = \{x,y\}$ and $\{x[t], y[t]\} = \{2\text{Cos}[t], 2\text{Sin}[t]\}$ a counterclockwise parameterization, I calculate:

$$\oint_C -n[x, y] dx + m[x, y] dy$$

$$= \int_0^{2\pi} (-n[x[t], y[t]] x'[t] + m[x[t], y[t]] y'[t]) dt = 8\pi,$$

thus the net flow across the curve is from inside to outside.

Measure by hand calculation the net flow along the circle

$$x^2 + y^2 = 9$$

of a fluid whose velocity is given by the vector field

$$\text{Field}[x, y] = \{-y, x\}.$$

Don't forget that $\text{Sin}[t]^2 + \text{Cos}[t]^2 = 1$.

Is the net flow of $F[x, y]$ along this curve clockwise or counterclockwise?

For $\{m[x,y], n[x,y]\} = \{x,y\}$ and $\{x[t], y[t]\} = \{2\text{Cos}[t], 2\text{Sin}[t]\}$ a counterclockwise parameterization, I calculate:

$$\oint_C m[x, y] dx + n[x, y] dy$$

$$= \int_0^{2\pi} (m[x[t], y[t]] x'[t] + n[x[t], y[t]] y'[t]) dt = 0,$$

thus the net flow along the circle is 0 (neither clockwise nor counterclockwise).

□ R.22)

Suppose C_1 and C_2 are physically the same curve, but they are parameterized so that the starting point of C_1 is the ending point of C_2 , and the ending point of C_1 is the starting point of C_2 .

Express

$$\int_{C_1} m[x, y] dx + n[x, y] dy$$

in terms of

$$\int_{C_2} m[x, y] dx + n[x, y] dy.$$

$$\int_{C_1} m[x, y] dx + n[x, y] dy = -\int_{C_2} m[x, y] dx + n[x, y] dy.$$

□ R.23)

Take a given vector field

$$\text{Field}[x, y] = \{m[x, y], n[x, y]\}.$$

If C_1 and C_2 are two curves that are parameterized to start at a given point $\{a, b\}$, and end at another given point $\{c, d\}$, but are not the same physical curve, what do you look for to determine whether you guaranteed that

$$\int_{C_1} m[x, y] dx + n[x, y] dy = \int_{C_2} m[x, y] dx + n[x, y] dy?$$

I would check to see if $\text{Field}[x, y]$ is a gradient field. If a vector field

$$\text{Field}[x, y] = \{m[x, y], n[x, y]\}$$

is a gradient field, then for any curve C , the value of the path integral

$$\int_C m[x, y] dx + n[x, y] dy$$

depends on the location of the starting point of C and the location of the end point of C but does not depend on the specific path C takes as it runs from its start to its end.

□ R.24)

Here are four vector fields:

$$\text{Field1}[x, y] = \{y e^{xy}, x e^{xy}\},$$

$$\text{Field2}[x, y] = \{y, -x\},$$

$$\text{Field3}[x, y] = \{\sin[2x], \cos[5y]\}, \text{ and}$$

$$\text{Field4}[x, y] = \{3x - 2y, -2x + 5y\}.$$

All but one of these vector fields are gradient fields.

Identify the oddball.

$\text{Field2}[x, y] = \{y, -x\}$ is the only field which does not pass the gradient test, since

$D[m[x,y], y] = 1$ which is not the same as $D[n[x,y], x] = -1$.

□ R.25)

Does

$$\text{Field}[x, y] = \{y \text{Sin}[x y], x \text{Sin}[x y]\}$$

pass the gradient test?

Does

$$\text{Field}[x, y] = \{x \text{Sin}[x y], y \text{Sin}[x y]\}$$

pass the gradient test?

The first field passes the gradient test, with $D[m[x,y],y] = \text{Sin}[xy] + xy \text{Cos}[xy] = D[n[x,y],x]$ and neither function has singularities. The second field does not pass the gradient test since $D[m[x,y],y] = x^2 \text{Cos}[xy]$, and $D[n[x,y],x] = y^2 \text{Cos}[xy]$.

□ R.26)

Go with

$$\text{Field}[x, y] = \{e^x \text{Sin}[y], e^x \text{Cos}[y]\}$$

and calculate $\text{divField}[x, y]$.

What does your result tell you about the net flow of $\text{Field}[x, y]$

across the ellipse $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{7}\right)^2 = 1$?

Since there are no singularities inside the ellipse,

$\text{divField}[x, y] = D[m[x, y], x] + D[n[x, y], y] =$

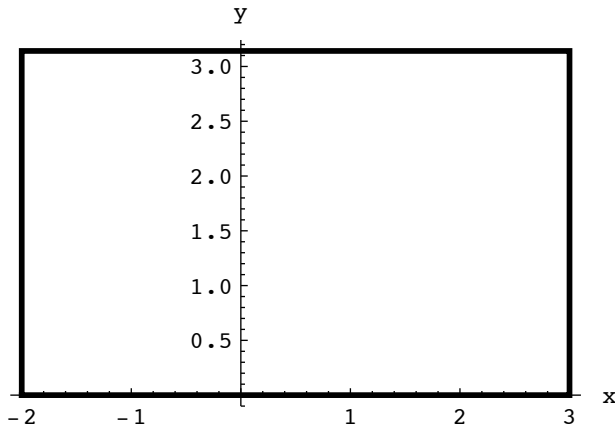
$e^x \text{Sin}[y] + -e^x \text{Sin}[y] = 0$.

This tells me that there is no net flow across the ellipse.

□ R.27)

Here's the rectangle R with corners at

$\{-2, 0\}$, $\{3, 0\}$, $\{3, \pi\}$ and $\{-2, \pi\}$:



Use a 2D integral to measure the net flow of the vector field

$$\text{Field}[x, y] = \{x \cos[y], e^{-x^2/2} + y^2\}$$

across the boundary curve C of this rectangle.

Say why you are happy to make this measurement by calculating your 2D integral instead of making this measurement by calculating the path integral

$$\oint_C -(e^{-x^2/2} + y^2) dx + x \cos[y] dy$$

$$\text{divField}[x,y] = \cos[y] + 2y$$

$$\text{Thus we calculate: } \int_0^{\pi} \int_{-2}^3 \text{divField}[x, y] dx dy = \int_0^{\pi} 5 \cos[y] + 10 y dy = 5 \pi^2.$$

The flow across the curve C is from inside to outside.

This is a much simpler calculation than that crazy integral above, that's for sure!!

□ R.28)

Identify the points $\{x, y\}$ of the vector field

$$\text{Field}[x, y] = \{-x^2 + y, y + 2 \sin[\pi x]\}$$

that are sources of new fluid.

Identify the points $\{x, y\}$ of the vector field

$$\text{Field}[x, y] = \{-x^2 + y, y + 2 \sin[\pi x]\}$$

that are sinks for old fluid.

$\text{divField}[x,y] = -2x+1$. Thus points $\{x,y\}$ with x less than $1/2$ are sources, and points $\{x,y\}$ with x greater than $1/2$ are sinks. (check the sign of divField).

□ R.29)

Given a vector field

$$\text{Field}[x, y] = \{m[x, y], n[x, y]\}$$

with the extra property that

$$\text{divField}[x, y] = D[m[x, y], x] + D[n[x, y], y] > 0$$

for every x and y , explain how you know that the net flow of

$$\text{Field}[x, y] = \{m[x, y], n[x, y]\}$$

across any closed curve without loops is from inside to outside.

What happens in the case that $\text{divField}[x, y] < 0$ for every x and y ?

What happens in the case that $\text{divField}[x, y] = 0$ for every x and y ?

I suppose we assume no singularities here.

If $\text{divField}[x, y]$ is always positive, then every point is a source, thus the net flow across any closed curve without loops is going to be from inside to outside. (If can only flow from outside to inside if there is at least one sink)

If divField is always negative, then similarly, the flow will be from outside to inside since everything is a sink.

If divField is always 0, then the net flow is always 0 since there are no sources or sinks.