Statement on Research

Melissa A. Dennison

My main research interests lie in number theory. Although my dissertation research has focused on combinatorial number theory, I have also spent time studying topics in number theory, analytic number theory, and analysis. Additionally, my research is tied to generating functions and graph theory.

Due to the nature of the subject, my research lends itself particularly well to undergraduate research. I have several ideas suitable for undergraduate research projects and I look forward to mentoring undergraduate research in the future.

1. Introduction

In 1858, the mathematician M. A. Stern [11] studied the so-called "diatomic array of integers", which was motivated by a function studied by Eisenstein. The diatomic array can be completely understood by consideration of the Stern sequence. The *Stern sequence* is defined as follows:

$$s(0) = 0, \quad s(1) = 1; \tag{1.1}$$

$$s(2n) = s(n), \quad \text{for } n \geq 1; \tag{1.2}$$

$$s(2n+1) = s(n) + s(n+1), \quad \text{for } n \geq 1. \tag{1.3}$$

Here is a table of the first 17 entries in the Stern sequence:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$s(n)$	0	1	1	2	1	3	2	3	1	4	3	5	2	5	3	4	1

Table 1. The Stern sequence

De Rham [3] was the first to consider the Stern sequence as defined above. He attributed the name to Bachmann [2], who considered only the diatomic array. The related *Stern-Brocot array* [5] was used in defining Minkowski's **?**-function [7], and the Stern sequence has recently been used to understand 2-regular sequences [1] and the Tower of Hanoi graph [6].

In my research, I define a new family of sequences which have the opposite recursion as the Stern sequence. The *general bow sequence* is defined as follows:

$$b_{\alpha,\beta}(0) = 0, \quad b_{\alpha,\beta}(1) = \alpha, \quad b_{\alpha,\beta}(2) = \beta; \tag{1.4}$$

$$b_{\alpha,\beta}(2n) = b_{\alpha,\beta}(n) + b_{\alpha,\beta}(n+1), \quad \text{for } n \geq 2; \tag{1.5}$$

$$b_{\alpha,\beta}(2n+1) = b_{\alpha,\beta}(n), \quad \text{for } n \geq 1. \tag{1.6}$$

Here is a table of the first 10 entries in the general bow sequence:

n	0	1	2	3	4	5	6	7	8	9
$b_{\alpha,\beta}(n)$	0	α	β	α	$\alpha+\beta$	β	$2\alpha+\beta$	α	$\alpha+2\beta$	$\alpha+\beta$

Table 2. The general bow sequence

Many of the properties of the bow sequences are related to known properties of the Stern sequence. Let us first consider a few of the important properties of the Stern sequence.

2. Background on the Stern Sequence

First, we consider the generating function for the Stern sequence.

Theorem 2.1. [10] *The generating function for the Stern sequence is given by the following:*

$$S(x) := \sum_{n=0}^{\infty} s(n)x^n = x\prod_{j=0}^{\infty}(1 + x^{2^j} + x^{2\cdot 2^j}). \tag{2.1}$$

We interpret the above statement combinatorially in the following remark.

Remark. *Combinatorially, the generating function tells us that $s(n)$ is the number of ways of writing $n-1$ as $\sum_{i\geq 0} c_i 2^i$ where $c_i \in \{0,1,2\}$.*

We can also use the generating function to determine the behavior of the Stern sequence modulo two. With a few quick calculations, we find that:

$$S(x) = x\prod_{j=0}^{\infty}(1 + x^{2^j} + x^{2\cdot 2^j}) \equiv \frac{x+x^2}{1-x^3} \pmod 2. \tag{2.2}$$

Thus we have the following theorem:

Theorem 2.2. [9], [10] *For $n \geq 0$, $s(n)$ is even exactly when $3 \mid n$.*

I obtain similar results for the bow sequences.

3. Selected Results

First, for the general bow sequence, I obtained the following generating function:

Theorem 3.1. *For the general bow sequence,*

$$B_{\alpha,\beta}(x) = \alpha x + \alpha x^3 + \alpha x^2 \sum_{k=0}^{\infty} x^{2\cdot 2^k} \prod_{j=0}^{k}(1 + x^{2\cdot 2^j} + x^{3\cdot 2^j}) + \beta x^2 \prod_{k=0}^{\infty}(1 + x^{2\cdot 2^k} + x^{3\cdot 2^k}). \tag{3.1}$$

I derived this formula by separating the sum into even and odd terms, applying the recurrence, and applying Lebesgue's Dominated Convergence Theorem. Thus, for the case $\{\alpha,\beta\} = \{0,1\}$ I have the following formula:

$$B_{0,1}(x) = x^2 \prod_{k=0}^{\infty}(1 + x^{2\cdot 2^k} + x^{3\cdot 2^k}). \tag{3.2}$$

This statement is interpreted as:

Corollary 3.2. *Combinatorially, $b_{0,1}(n)$ is the number of ways of writing $n-2$ as the sum $\sum_i c_i 2^i$ where $c_i \in \{0,2,3\}$.*

Similarly, for the case $\{\alpha,\beta\} = \{1,0\}$, I have the following formula.

$$B_{1,0}(x) = x + x^3 + x^2 \sum_{k=0}^{\infty} x^{2\cdot 2^k} \prod_{j=0}^{k}(1 + x^{2\cdot 2^j} + x^{3\cdot 2^j}). \tag{3.3}$$

We can interpret this statement as:

Corollary 3.3. *Combinatorially, $b_{1,0}(n)$ is the number of ways of writing $n-2-2^{k+1}$ as the sum $\sum_{i=0}^{k} c_i 2^i$ where $c_i \in \{0,2,3\}$ and $k \in \mathbb{N}$.*

Remark. *By rearranging, I also found that $b_{1,0}(n)$ is also the number of ways of writing $n-2$ as the sum $\sum_{i=0}^{k} c_i 2^i$ where $k \in \mathbb{N}$, $c_i \in \{0,2,3\}$ for $i \leq k-1$ and $c_k \in \{2,4,5\}$.*

We would also like to consider the bow sequences modulo two. First, note from the recursion that $b_{\alpha,\beta}(n)$ is linear in α and β. Accordingly, we can write:

$$b_{\alpha,\beta}(n) = \alpha b_{1,0}(n) + \beta b_{0,1}(n). \tag{3.4}$$

Thus it is enough to consider $b_{1,0}(n)$ and $b_{0,1}(n)$ modulo two. Define $v_{\alpha,\beta}(n)$ as follows:

$$v_{\alpha,\beta}(n) := \big(b_{\alpha,\beta}(7n), b_{\alpha,\beta}(7n+1), b_{\alpha,\beta}(7n+2), ..., b_{\alpha,\beta}(7n+6)\big) \pmod 2. \tag{3.5}$$

Interestingly, there is only one case for $v_{0,1}(n)$.

Theorem 3.4. *For $b_{0,1}(n)$ and all n,*

$$v_{0,1}(n) \equiv v^0(n) := (0,0,1,0,1,1,1) \pmod 2. \tag{3.6}$$

I show this is true by proving the following:

$$B_{0,1}(x) = x^2 \prod_{k=0}^{\infty}(1 + x^{2\cdot 2^k} + x^{3\cdot 2^k}) \equiv \frac{x^2+x^4+x^5+x^6}{1-x^7} \pmod 2. \tag{3.7}$$

However, for $b_{1,0}(n)$ the cases are more complicated. Define J_m as follows:

$$J_m := \Big(\tfrac{3}{7}(2^m - 1), \tfrac{3}{7}(2^{m+1}-1)\Big] \cap \mathbb{Z} \tag{3.8}$$

Then I have the following theorem.

Theorem 3.5. *For $b_{1,0}(n)$ we have four cases modulo two:*

$$v_{1,0}(n) \equiv \begin{cases} v^1(n) = (0,\ 1,\ 0,\ 1,\ 1,\ 1,\ 0) & \text{for } n \in J_{3r+2} \setminus \big\{\tfrac{3}{7}(2^{3r+3}-1)\big\},\ \text{for } r \geq 0, \\ v^{1*}(n) = (0,\ 1,\ 0,\ 1,\ 1,\ 0,\ 0) & \text{for } n = \tfrac{3}{7}(2^{3r}-1),\ \text{for } r \geq 0, \\ v^2(n) = (1,\ 0,\ 1,\ 1,\ 1,\ 0,\ 0) & \text{for } n \in J_{3r},\ \text{for } r \geq 1, \\ v^3(n) = (1,\ 1,\ 1,\ 0,\ 0,\ 1,\ 0) & \text{for } n \in J_{3r+1},\ \text{for } r \geq 0. \end{cases}$$

In general, I found that $\frac{3}{7}$ of the terms are even for a given pair $\{\alpha, \beta\}$ where $gcd(\alpha,\beta)$ is odd. Clearly, in the case where $gcd(\alpha,\beta)$ is even, all of the terms in the bow sequence are even.

Theorem 3.4 and Theorem 3.5 lead to another very nice theorem. Define m to be *purely even* if $b_{0,1}(m)$ and $b_{1,0}(m)$ are both even.

Theorem 3.6. *Each 7-tuple $\{7n,\ 7n+1,\ ...,\ 7n+6\}$ contains exactly one purely even term.*

I prove this theorem by examining the cases in Theorems 3.4 and 3.5, and then go on to state exactly where these purely even terms occur.

4. Future Work

The previous discussion provokes several questions:

Question 1. *Given $A \subset \mathbb{Z}$, what can we say about the function which counts the number of ways you can write $n = \sum_{i=0}^{k} c_i 2^i$, for $c_i \in A$?*

We have already seen that the subsets $A_0 = \{0,1,2\}$, $A_1 = \{0,2,3\}$ correspond to $s(n+1)$, and $b_{0,1}(n+2)$, respectively. I have also proven that $b_{1,0}(n+3)$ is the number of ways of writing n as the sum $\sum_{i=0}^{k} c_i 2^i$, for $c_i \in A_2 = \{1,3,4\}$. Additionally, the sets $A_d = \{0, 1, \cdots d-1\}$ have been studied by Euler (d=2) [4] and Reznick [8]. By studying another related generating function, I have discovered and proved a similar relation for $A_3 = \{0,1,3\}$, but I hope to prove many more such relations by considering generating functions of the following form:

$$G^{a_1,a_2,a_3,\ldots,a_m}(x) := x\prod_{j=0}^{\infty}(1 + x^{a_1\cdot 2^j} + x^{a_2\cdot 2^j} + \cdots + x^{a_m\cdot 2^j}) \tag{4.1}$$

Question 2. *When is $b_{\alpha,\beta} \equiv 0,\ 1,\ \cdots k-1$ modulo k for $k \geq 3$?*

I have numerical evidence for a conjecture for the distribution of terms equivalent to m modulo k with $\gcd\{\alpha, \beta\} = 1$, and $k \leq 10$. Thus far, applying graph theory to triples has yielded good results with $k = 2$ and 3, and I hope to generalize these results for higher values of k.

References

[1] J.-P. Allouche, J. Shallit, The Ring of k-Regular Sequences, *Theoret. Comput. Sci.*, **98** (1992), 163-197.
[2] P. Bachmann, Niedere Zahlentheorie, v. 1, Leipzig 1902, Reprinted by Chelsea, New York, 1968.
[3] G. de Rham, Un peu de mathématics à propos d'une coube plane, *Elemente de Math.*, **2** (1947), 73-76, 89-97.
[4] L. Euler, Introduction in analysin infinitorum. Lausanne, 1748, *in Opera Omnia Series Prima Opera Math.* vol. 8, B. G. Teubner, Leipzig, 1922.
[5] R. L. Graham, D. E. Knuth, O. Patashnik, *Concrete Mathematics*, 2nd ed., Addison-Wesley, Boston, MA, 1994.
[6] A. Hinz, S. Klavzar, U. Milutinović, D. Parisse, C. Petr, Metric Properties of the Tower of Hanoi Graphs and Stern's Diatomic Sequence, *Euro. J. Comb.*, **26** (2005), 693-708.
[7] H. Minkowski, Zur Geometrie der Zahlen, Ver. III, Int. Math-Kong., Heidelberg 1904, 164-173; in *Gesammelte Abhandlungen, Vol. 2*, Chelsea, New York (1967), 45-52.
[8] B. Reznick, Some digital partition functions, in: B. C. Berndt et al. (Eds.), Analytic Number Theory, Proceedings of a Conference in Honor of Paul T Bateman, Birkhäuser, Boston, (1990), 451-477.
[9] _____, classnotes, Stern Sequences course, University of Illinois at Urbana-Champaign, Spring 2006.
[10] _____, Regularity Properties of the Stern Enumeration of the Rationals, *Journal of Integer Sequences*, **11** (2008).
[11] M. A. Stern, Ueber eine Zahlentheoretische Funktion, *J. Reine Angew. Math.*, **55** (1858), 193-220.

Department of Mathematics, University of Illinois, Urbana, IL 61801
E-mail address: masimmon@math.uiuc.edu